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A quantum mechanical approach to a fuzzy theory

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Abstract

Recently, we proposed a general measurement theory for classical and quantum systems (i.e., "objective fuzzy measurement theory"). In this paper, we propose "subjective fuzzy measurement theory", which is characterized as the statistical method of the objective fuzzy measurement theory. Our proposal of course has a lot of advantages. For example, we can directly see "membership functions" (= "fuzzy sets") in this theory. Therefore, we can propose the objective and the subjective methods of membership functions. As one of the consequences, we assert the objective (i.e., individualistic) aspect of Zadeh's theory. Also, as a quantum application, we clarify Heisenberg's uncertainty relation. © 1997 Elsevier Science B.V.

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1. Introduction and fuzzy measurement theory

In this paper, we propose "subjective fuzzy measurement theory", which is characterized as the statistical method of "(objective) fuzzy measurement theory" (cf. [9]).

Our original motivation (i.e., the first purpose of this paper) is to clarify Heisenberg's uncertainty relation. Recently, in a series of our papers [6–8, 10], we investigated and proposed the mathematical representation of Heisenberg's uncertainty relation. Let A_1 and A_2 be physical quantities (i.e., self-adjoint operators) in a Hilbert space V, in which a quantum system is formulated. The quartet $[\hat{A}_1, \hat{A}_2, U, \phi_0]$ is called an *approximate simultaneous measurement* of A_1 and A_2, if it satisfies the following conditions (i)–(iv): (i) U is a Hilbert space and ϕ_0 is an element in U such that $||\phi_0||_U = 1$, (ii) the self-adjoint operators $\hat{A}_1$ and $\hat{A}_2$ in a tensor Hilbert space $V \otimes U$ commute, (iii) $\langle \psi \otimes \phi_0, \hat{A}_k(\psi \otimes \phi_0)\rangle_{V\otimes U} = \langle \psi, A_k\psi\rangle_V$ ($\forall\psi \in \mathrm{Dom}(A_k)$, the domain of A_k, $k = 1,2$), (iv) $\hat{A}_k$ and $A_k \otimes I$ commute ($k = 1,2$).

Then, we have the following proposition (cf. [6]).

Proposition 1.1 (Heisenberg's uncertainty relation). *Let A_1 and A_2 be physical quantities in a Hilbert space V. Then, the following statements* (A) *and* (B) *hold*:

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(A) *There exists an approximate simultaneous measurement* $[\hat{A}_1, \hat{A}_2, U, \phi_0]$ *of* A_1 *and* A_2. *Furthermore, for any positive* ε, *we can take this* $[\hat{A}_1, \hat{A}_2, U, \phi_0]$ *such that*

$$
\begin{aligned}
&||(\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)||_{V\otimes U} \\
&= \begin{cases} \varepsilon ||A_1\psi||_V & \text{for all } \psi \in Dom(A_1) \text{ such that } ||\psi||_V = 1 \quad (\text{if } k = 1), \\ \varepsilon^{-1} ||A_2\psi||_V & \text{for all } \psi \in Dom(A_2) \text{ such that } ||\psi||_V = 1 \quad (\text{if } k = 2). \end{cases}
\end{aligned}
$$

(B) *However, for any approximate simultaneous measurement* $[\hat{A}_1, \hat{A}_2, U, \phi_0]$ *of* A_1 *and* A_2, *the following inequality (Heisenberg's uncertainty relation) holds*:

$$
\begin{aligned}
&||(\hat{A}_1 - A_1 \otimes I)(\psi \otimes \phi_0)||_{V\otimes U} \cdot ||(\hat{A}_2 - A_2 \otimes I)(\psi \otimes \phi_0)||_{V\otimes U} \\
&\geqslant |\langle A_1\psi, A_2\psi\rangle_V - \langle A_2\psi, A_1\psi\rangle_V|/2
\end{aligned}
$$

for all $\psi \in Dom(A_1) \cap Dom(A_2)$ *such that* $||\psi||_V = 1$.

When A_1 *and* A_2 *is a position quantity and a momentum quantity, respectively (i.e.,* $A_1A_2 - A_2A_1 = i\hbar, \hbar$ *is the Plank constant), the right-hand side of this inequality is of course equal to* $\hbar/2$.

This is the mathematical representation of Heisenberg's uncertainty relation. As an immediate consequence of Proposition 1.1, we solved the paradox between Einstein–Podolsky–Rosen experiment and Heisenberg's uncertainty relation. Though this was immediately accepted by many physicists, we do not think that the above Proposition 1.1 is the final version of Heisenberg's uncertainty relation. That is because the meaning of the quantities $||(\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)||_{V\otimes U}$, $k = 1,2$, is not yet clear (though it may be considered as "error"). Therefore, our first purpose is to answer the general question "What is a measurement error ?". Of course this should be answered for all measurements (i.e., classical and quantum measurements).

On the other hand, recently in [9], we proposed a foundation of measurements (i.e., "(objective) fuzzy measurement theory"), which is characterized as a kind of generalization of Born's quantum measurement theory. Thus, this theory is a general measurement theory for classical and quantum systems. And furthermore, we proposed the identification: "measurement" = "inference". As one of the consequences, we showed that the standard syllogism (i.e., $A \Rightarrow B, B \Rightarrow C$ implies $A \Rightarrow C$) is true for classical systems. This is of course quite important. That is because "symbolic logic" is merely a mathematical rule, and therefore, the justification of the "symbolic logic" for this real world should be guaranteed by a certain principle. In fact, the standard syllogism does not always hold for quantum systems. Also, in [9] we mentioned, as a short sketch, that other "fuzzy theories in the wide sense" (i.e., "Zadeh's fuzzy sets theory", "Kolmogorov's probability theory", "Bayesian statistics" and so on.) should be characterized as some aspects of fuzzy measurement theory. That is because we consider that Born's assertion (i.e., quantum theory) is most fundamental, and therefore others are "methods" (or "mathematics"). Thus, the second purpose of this paper is to examine this sketch precisely.

The above first and second purposes are of course closely connected with each other. In order to characterize "measurement error", we must provide a lot of preparations in the fuzzy measurement theory. That is, we must construct "subjective fuzzy measurement theory". This construction is almost equal to examining the above sketch. Therefore, we can also say that our main purpose of this paper is to propose "subjective fuzzy measurement theory".

In Section 1 and the next Section 2 we review the fuzzy measurement theory (proposed in [9]) and study the objective (or, individualistic) aspect of this theory. In Sections 3 and 4 we propose "subjective fuzzy measurement theory", which is characterized as a "statistical" method of "(objective) fuzzy measurement theory". Thus, this subjective theory can always be interpreted from the objective point of view. Throughout these arguments, we clarify the relation between "objective fuzzy measurement theory" (with C^*-algebraic formulation) and "subjective fuzzy measurement theory" (with W^*-algebraic formulation). Also we define "measurement error" in a quite general situation under the W^*-algebraic formulation. Compared to Kolmogorov's probability theory, our proposal has a lot of advantages. Of course, its "objectivity" (i.e., "reality") is most essential. Also, for example, we can directly see "membership functions" (="fuzzy sets") in this theory. Therefore, in Section 5 [resp. Section 6], we can introduce the subjective [resp. objective] method of membership functions. And we show that these methods provide precise analysis to "fuzziness". As one of the conclusions, we formulate "Shannon's entropy" and "Bayes's postulate" in this subjective theory. Also, we can emphasize the objective (or, individualistic) aspect of "Zadeh's fuzzy sets theory". Lastly, as a quantum application, in Section 7 we clarify "measurement error" in Heisenberg's uncertainty relation.

According to [9], we now introduce "(objective) fuzzy measurement theory". In order to propose "subjective fuzzy measurement theory" (in Sections 3 and 4), we must completely understand this objective theory. This theory is formulated in terms of C^*-algebras (cf.[15, 9]). (In this paper we mainly devote ourselves to commutative C^*-algebras, so the deep knowledge of C^*-algebras is not needed.) Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{A}^*$ be the dual Banach space of $\mathscr{A}$. That is, $\mathscr{A}^* \equiv \{\rho : \rho$ is a continuous linear functional on $\mathscr{A}\}$ with the norm $||\cdot||_{\mathscr{A}^*}$ (i.e., $||\rho||_{\mathscr{A}^*} \equiv \sup\{|\rho(T)|: ||T||_{\mathscr{A}} \leqslant 1\}$). The linear functional $\rho(T)$ is sometimes denoted by ${}_{\mathscr{A}^*}\langle \rho, T\rangle_{\mathscr{A}}$. Define the *mixed-state class* $\mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$ such that $\mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*) \equiv \{\rho \in \mathscr{A}^*: ||\rho||_{\mathscr{A}^*} = 1$ and $\rho \geqslant 0$ (i.e., $\rho(T^*T) \geqslant 0$ for all $T \in \mathscr{A}$)$\}$. A mixed state ρ^{p} (i.e., $\rho^{\mathrm{p}} \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$) is called a *pure state* if it satisfies that "$\rho^{\mathrm{p}} = \lambda\rho_1 + (1-\lambda)\rho_2$ (for some $\rho_1, \rho_2 \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$ and $0 < \lambda < 1$)" implies "$\rho^{\mathrm{p}} = \rho_1 = \rho_2$". Define $\mathfrak{S}^{\mathrm{p}}(\mathscr{A}^*) \equiv \{\rho^{\mathrm{p}} \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*) : \rho^{\mathrm{p}}$ is a pure state $\}$. An element T ($\in \mathscr{A}$) is called *positive* (and denoted by $T \geqslant 0$) if there exists an element $T_0(\in \mathscr{A})$ such that $T = T_0^* T_0$. Also, a positive element T is called a *projection* if $T = T^2$ holds.

Remark 1.2. In general, a C^*-algebra $\mathscr{A}$ has no identity element I, i.e., $T = IT = TI (\forall T \in \mathscr{A})$. In this case, we can easily construct the C^*-algebra $\bar{\mathscr{A}}$ with the identity element I as the C^*-algebra generated by $\{T + \alpha I: T \in \mathscr{A}, \alpha \in \mathbf{C}$, i.e., α is a complex number$\}$ with the norm $||T+\alpha I||_{\bar{\mathscr{A}}} = \sup\{||TS + \alpha S||_{\mathscr{A}}: ||S||_{\mathscr{A}} = 1\}$. Thus, it holds that ${}_{\mathscr{A}^*}\langle \rho, T + \alpha I\rangle_{\bar{\mathscr{A}}} = {}_{\mathscr{A}^*}\langle \rho, T\rangle_{\mathscr{A}} + \alpha||\rho||_{\mathscr{A}^*}$ for all ρ ($\in \mathscr{A}^*$) such that $\rho \geqslant 0$.

Now we provide the elementary examples (i.e., $C_0(\Omega), C(\Omega)$ and $\mathscr{C}(V)$) of C^*-algebras, which will be used frequently in this paper. Thus, these examples should also be regarded as "notations".

Example 1.3 ($C_0(\Omega)$ *and* $\mathscr{C}(V)$). (i) Let $C_0(\Omega)$ be the algebra, under pointwise multiplication, of all complex valued, continuous functions on a locally compact Hausdorff space Ω that vanish at infinity. Define the norm $||f||_{C_0(\Omega)} = \max_{\omega\in\Omega} |f(\omega)|$, and $f^*(\omega) = \overline{f(\omega)}$, i.e., the complex conjugate. Then, $\mathscr{A} \equiv C_0(\Omega)$ is a commutative C^*-algebra, that is, $f_1 f_2 = f_2 f_1$ holds for any $f_1, f_2 (\in C_0(\Omega))$. If Ω is compact, the $C_0(\Omega)$ is often denoted by $C(\Omega)$. Here note that "Ω is compact" $\Leftrightarrow$ "$C_0(\Omega)$ has the identity I". It is well known that $\overline{C_0(\Omega)}(\equiv \bar{\mathscr{A}}) = C(\Omega \cup \{\infty\})$, where $\Omega \cup \{\infty\}$ is the one point compactification of Ω. Also, Gelfand theorem says that any commutative C^*-algebra $\mathscr{A}$ can be identified with some $C_0(\Omega)$.

(ii) Put $B(V) = \{T$: T is a bounded linear operator from a Hilbert space V into itself$\}$. Define $||T||_{B(V)} = \sup\{||T\psi||_V: ||\psi||_V \leqslant 1\}$, and $(T_1T_2)(\psi) = T_1(T_2\psi) (\forall \psi \in V)$. And T^* is the adjoint operator of T. This $B(V)$ is of course a non-commutative C^*-algebra. Here note that $\mathscr{C}(V) \equiv \{T \in B(V)$: T is a compact operator$\}$ is a C^*-subalgebra of $B(V)$. It is clear that "the dimension of V is finite" $\Leftrightarrow$ "$\mathscr{C}(V)$ has the identity I". Also note that $\overline{\mathscr{C}(V)} = \{T + \alpha I: T \in \mathscr{C}(V), \alpha \in \mathbf{C}\} \subseteq B(V)$.

Example 1.4 (*Continued from the above example*). (i) It is well-known (cf. [17]) that $C_0(\Omega)^* = \mathscr{M}(\Omega)$ and $\mathfrak{S}^{\mathrm{m}}(C_0(\Omega)^*) = \mathscr{M}_{+1}(\Omega)$. Here $\mathscr{M}(\Omega) = \{\mu : \mu$ is a regular signed measure on $\Omega\}$, and $\mathscr{M}_{+1}(\Omega) = \{\mu \in \mathscr{M}(\Omega)$: μ is non-negative and $\mu(\Omega) = 1\}$. Also we see that $\mathfrak{S}^{\mathrm{p}}(C_0(\Omega)^*) = \mathscr{M}^{\mathrm{p}}_{+1}(\Omega) \equiv \{\delta_{\omega_0} \in \mathscr{M}_{+1}(\Omega) : \omega_0 \in \Omega\}$, where δ_{ω_0} is a point measure at ω_0, i.e., $\delta_{\omega_0}(f) \equiv \int_\Omega f(\omega)\delta_{\omega_0}(d\omega) = f(\omega_0) (\forall f \in C_0(\Omega))$.

(ii) When $\mathscr{A} = \mathscr{C}(V)$, we see that $\mathscr{C}(V)^* = \mathrm{Tr}(V)$, the class of trace operators, and $\mathfrak{S}^{\mathrm{m}}(\mathscr{C}(V)^*) = \mathrm{Tr}_{+1}(V) \equiv \{\rho \in \mathrm{Tr}(V): \rho \geqslant 0, ||\rho||_{\mathrm{Tr}(V)} = 1\}$. Also, it is well known that "$\rho \in \mathfrak{S}^{\mathrm{p}}(\mathscr{C}(V)^*)$" $\leftrightarrow$ "there exists $\psi \in V (||\psi||_V = 1)$ such that $\rho = |\psi\rangle\langle\psi|$", where the $|\psi\rangle\langle\psi| (\in B(V))$ is defined by $(|\psi\rangle\langle\psi|)\phi = \langle\psi, \phi\rangle_V \psi$ for all $\phi \in V$.

The concept of "fuzzy observable" was first introduced in quantum mechanics for a W^*-algebra $B(V)$ (cf. Definition 4.2 in Section 4) by Davies [2]. Thus, the following definition (for fuzzy systems) is the C^*-algebraic version of his idea.

Definition 1.5 (*C^*-observable*). Let $\mathscr{A}$ be a C^*-algebra. A C^*-observable (or, continuous observable, fuzzy observable, *or in short*, observable) $\mathbf{O} \equiv (X, \mathscr{P}_0(X), F)$ in $\mathscr{A}$ is defined such that it satisfies that

(i) a label set X is a countable (or finite) set, and $\mathscr{P}_0(X) = \{\Xi \subseteq X : \Xi \text{ or } X \setminus \Xi \text{ is finite}\}$,

(ii) for every $\Xi \in \mathscr{P}_0(X)$, $F(\Xi)$ is a positive element in $\mathscr{A}$ (or precisely, in $\bar{\mathscr{A}}$, cf. Remark 1.2) such that $F(\emptyset) = 0$ and $F(X) = I$, where 0 is the 0-element and I is the identity element, and

(iii) for any countable decomposition $\{\Xi_1, \Xi_2, \ldots, \Xi_n, \ldots\}$ of Ξ, $(\Xi, \Xi_n \in \mathscr{P}_0(X))$, it holds that $\rho(F(\Xi)) = \lim_{N\to\infty} \rho(\sum_{n=1}^{N} F(\Xi_n))$ $(\forall \rho \in \mathscr{A}^*)$.

Also, if $F(\Xi)$ is a projection for every Ξ $(\in \mathscr{P}_0(X))$, a C^*-observable $(X, \mathscr{P}_0(X), F)$ is called a crisp C^*-observable.

Remark 1.6. The above condition (i) in Definition 1.5 may be weaken as "(i)$'$ $\mathscr{P}_0(X)$ is a subfield of the power set $\mathscr{P}(X) \equiv \{\Xi : \Xi \subseteq X\}$". Even under this weak condition (i)$'$, we can, by Hopf extension theorem (cf. [17]), get the probability measure space $(X, \bar{\mathscr{P}_0}(X), \rho^{\mathrm{m}}(F(\cdot)))$ for any $\rho^{\mathrm{m}} \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$, where $\bar{\mathscr{P}_0}(X)$ is the smallest σ-field that contains $\mathscr{P}_0(X)$. However, in this paper we assume the condition (i) in Definition 1.5. In particular, if X is finite, then $\mathscr{P}_0(X) = \mathscr{P}(X)$ clearly holds. We believe that, without loss of generality (or essence), we can assume that X is finite. Therefore, most arguments in this paper are treated under the condition that X is finite.

Let $(X, \mathscr{P}_0(X), F)$ be a C^*-observable in a C^*-algebra $\mathscr{A}$. Let g be a "measurable" map from X into Y (i.e., $g^{-1}(\Gamma) \in \mathscr{P}_0(X)$ for all $\Gamma \in \mathscr{P}_0(Y)$). Then, we can define the C^*-observable $(Y, \mathscr{P}_0(Y), G)$ in $\mathscr{A}$ such that $G(\Gamma) = F(g^{-1}(\Gamma))(\forall \Gamma \in \mathscr{P}_0(Y))$. This C^*-observable $(Y, \mathscr{P}_0(Y), G)(\equiv (Y, \mathscr{P}_0(Y), F(g^{-1}(\cdot))))$ is called the *image observable of g for $(X, \mathscr{P}_0(X), F)$*.

Example 1.7 (*Fuzzy numbers observable*). Let Δ be any positive number. Define the membership function (i.e., triangle fuzzy number) $\mathscr{Z}_\Delta (\in C_0(\mathbf{R})$, where $\mathbf{R}$ is the real line with the usual topology) such that

$$\mathscr{Z}_\Delta(\omega) = \begin{cases} 1 - \omega/\Delta, & 0 \leqslant \omega \leqslant \Delta, \\ \omega/\Delta + 1, & -\Delta \leqslant \omega \leqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Put $\mathbb{Z}_\Delta \equiv \{\Delta k : k \in \mathbb{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}\}$. Define the C^*-observable $\mathbf{O}_{\mathscr{Z}_\Delta} \equiv (\mathbb{Z}_\Delta, \mathscr{P}_0(\mathbb{Z}_\Delta), \zeta^\Delta_{(\cdot)})$ in the commutative C^*-algebra $C_0(\mathbf{R})$ such that $\zeta^\Delta_\Xi(\omega) = \sum_{x\in\Xi} \mathscr{Z}_\Delta(\omega - x)(\forall \Xi \in \mathscr{P}_0(\mathbb{Z}_\Delta), \forall \omega \in \mathbf{R})$. This C^*-observable is called a *fuzzy numbers observable* in $C_0(\mathbf{R})$. Putting $\Delta = 1$, we frequently use the fuzzy numbers observable $\mathbf{O}_{\mathscr{Z}} \equiv (\mathbb{Z}, \mathscr{P}_0(\mathbb{Z}), \zeta_{(\cdot)})$ in this paper.

Remark 1.8 (*Fuzziness*). It should be noted that $C_0(\mathbf{R})$ (or precisely, $\overline{C_0(\mathbf{R})}$) has only two projections, i.e., two constant functions 0 and 1. Thus, the concept "crisp observable" has no generality in $C_0(\mathbf{R})$. Therefore, the (objective) fuzzy measurement theory cannot start from "crispness" but "fuzziness". That is, "fuzziness" is essential for this theory. As seen later (in Section 4), "crispness" plays an important role in the W^*-algebraic formulation of (subjective) fuzzy measurement theory.

Remark 1.9 (*Fuzzy sets*). Let $\mathbf{O} \equiv (X = \{x_1, x_2, \ldots\}, \mathscr{P}_0(X), f_{(\cdot)})$ be a C^*-observable in a commutative C^*-algebra $C_0(\Omega)$. Note that the C^*-observable $\mathbf{O}$ can be identified with $\mathbf{D} \equiv \{f_{\{x_1\}}, f_{\{x_2\}}, \ldots\}$. That is because any $f_\Xi(\omega)$ is obtained by $f_\Xi(\omega) = \sum_{x\in\Xi} f_{\{x\}}(\omega)$. Also, membership functions $f_{\{x_n\}}$'s clearly satisfy that $0 \leqslant f_{\{x_j\}}(\omega) \leqslant 1$ and $\sum_{j=1}^{\infty} f_{\{x_j\}}(\omega) = 1(\forall \omega \in \Omega)$. Therefore, if we are allowed to use the word "fuzzy sets", the $\mathbf{D}$ is regarded as the "decomposition" of Ω by fuzzy sets $f_{\{x_n\}}$'s. That is, the C^*-observable $\mathbf{O}$

is equivalent to the "fuzzy decomposition" **D** of Ω. Though this view-point is interesting (and we often use "fuzzy sets"), our fuzzy theory is formulated by C^*-algebras and not "sets theory".

Now we can propose "(objective) fuzzy measurement theory" (cf. [9]). As the most basic requirement for a fuzzy theoretical description of a fuzzy system we have the following axiom:

Axiom 0 (Fuzzy system, state, observable, measurement, measured value, true value). *With any fuzzy system (or in short, system) S, a C^*-algebra $\mathscr{A}$ can be associated in which the fuzzy measurement theory of that system can be formulated.*

(i) *A state Θ of the fuzzy system S is represented by a pure state ρ^{p} ($\in \mathfrak{S}^{\mathrm{p}}(\mathscr{A}^*)$). And an observable $\mathcal{O}_X$(with a label set X) is represented by a C^*-observable $\mathbf{O} \equiv (X, \mathscr{P}_0(X), F)$ in the C^*-algebra $\mathscr{A}$. Also, the measurement $\mathscr{M}(\mathcal{O}_X, S_\Theta)$, i.e., the measurement of the observable $\mathcal{O}_X$ for the system S with the state Θ, is represented by $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$ in the C^*-algebra $\mathscr{A}$.*

(ii) *We can get a measured value x ($\in X$) by the measurement $\mathscr{M}(\mathcal{O}_X, S_\Theta)$.*

(iii) *Let $\mathcal{O}_Y$ be an observable, which is represented by the image observable $(Y, \mathscr{P}_0(Y), G)$ of g: $X \to Y$ for $\mathbf{O} \equiv (X, \mathscr{P}_0(X), F)$. (Here $\mathcal{O}_Y$ is also called an image observable of g for $\mathcal{O}_X$.) When we get the measured value x by the measurement $\mathscr{M}(\mathcal{O}_X, S_\Theta)$, we consider that the value (or, true value) of $\mathcal{O}_Y$ (for the system S with the state Θ) is equal to $g(x)$.*

Another axiom presented below is analogous to (or, a kind of generalizations of) Born's probabilistic interpretation of quantum mechanics.

Axiom 1 (Measurement axiom). *Consider a measurement $\mathscr{M}(\mathcal{O}_X, S_\Theta)$, which is represented by $\mathbf{M}_{\mathscr{A}}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), F), S_{\rho^{\mathrm{p}}})$ in a C^*-algebra $\mathscr{A}$. Assume that x ($\in X$) is the measured value obtained by the measurement $\mathscr{M}(\mathcal{O}_X, S_\Theta)$. Then, it holds that*

($*$) *the probability that the x ($\in X$) belongs to a set Ξ ($\in \mathscr{P}_0(X)$) is given by $\rho^{\mathrm{p}}(F(\Xi))(= {}_{\mathscr{A}^*}\langle \rho^{\mathrm{p}}, F(\Xi)\rangle_{\hat{\mathscr{A}}}$, or precisely, $= {}_{\mathscr{A}^*}\langle \rho^{\mathrm{p}}, F(\Xi)\rangle_{\bar{\mathscr{A}}}$).*

(For the reason that we use the word "probability", we will provide the precise arguments in Section 2.)

The (objective) fuzzy measurement theory is completely characterized by the two axioms (i.e., Axioms 0 and 1) described above. From now on, we often identify $\mathscr{M}(\mathcal{O}_X, S_\Theta)$ with $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$. We hope that the reader does not confuse $\mathscr{M}(\mathcal{O}_X, S_\Theta)$ with $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$. It is obvious that we can take an actual measurement $\mathscr{M}(\mathcal{O}_X, S_\Theta)$ even if we do not know its mathematical representation $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$.

Remark 1.10 (*Commutative fuzzy theory*). When $\mathscr{A} = \mathscr{C}(V)$, Axiom 1 is just Born's axiom in quantum mechanics. (Also, see Remark 4.6.) We introduce the following classification in fuzzy measurement theory:

$$\text{fuzzy measurement theory} \begin{cases} \text{commutative fuzzy theory (for classical systems),} \\ \text{non-commutative fuzzy theory (for quantum systems),} \end{cases}$$

where a C^*-algebra $\mathscr{A}$ is commutative or non-commutative. In this paper, we mainly devote ourselves to commutative fuzzy theory. The relation between "classical" and "quantum" is not simple in general. There may be an opinion that a principle should not exist for classical systems. For example, Newtonian equation is not a principle because it should be derived from Schrödinger equation. Similarly, Axiom 1 for $C_0(\Omega)$ should be derived from Axiom 1 for $\mathscr{C}(V)$. (We also consider that it may be possible under some restricted conditions.) And therefore, some may consider that commutative fuzzy theory is not needed. Though this opinion has a reason in some sense, we consider that a classical fuzzy system should start from Axiom 1 for a commutative C^*-algebra $C_0(\Omega)$. Therefore, we may agree to the opinion that the commutative fuzzy theory is not "ultimately objective".

The following example is one of typical classical measurements. Though it is quite simple, we believe that it does not miss the essences of classical measurements. In this paper, we will state several variants of this example (i.e., Remark 3.9, Examples 3.11, 4.9, 4.13 and 5.1). Also, for interesting examples of quantum measurements, see [5].

Example 1.11 (*The measurement of pencil's length*). We investigate the measurement of the length of a pencil. Let S be a system of a pencil with the length l_0 (cm), for example, $l_0 = 10\sqrt{2} = 14.14213\ldots$ Therefore, we assume that the system S is formulated in the commutative C^*-algebra $\mathscr{A} \equiv C_0(\mathbf{R})$. Also, the pure state of the system S is represented by the point measure δ_{l_0} $(\in \mathscr{M}^{\mathrm{p}}_{+1}(\mathbf{R}))$ on $\{l_0\}$. Let $\mathbf{O}_{\mathscr{Z}} \equiv (\mathbb{Z}, \mathscr{P}_0(\mathbb{Z}), \zeta_{(\cdot)})$ be the fuzzy numbers observable in $C_0(\mathbf{R})$ (cf. Example 1.7). Now consider the measurement $\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}})$ in $C_0(\mathbf{R})$, that is, the measurement of the fuzzy numbers observable $\mathbf{O}_{\mathscr{Z}}$ for the system S with the pure state δ_{l_0}. Here we see, by Axiom 1, that

(∗) the probability that the measured value z $(\in \mathbb{Z})$ belongs to Ξ $(\in \mathscr{P}_0(\mathbb{Z}))$ is given by

$$ {}_{\mathscr{M}(\mathbf{R})}\langle \delta_{l_0}, \zeta_{\Xi} \rangle_{\overline{C_0(\mathbf{R})}} = \int_{\Omega} \zeta_{\Xi}(\omega)\delta_{l_0}(\mathrm{d}\omega) = \zeta_{\Xi}(l_0). \tag{1.1} $$

A simple calculation shows that, for any $x(\in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots .\})$,

$$ (1.1) = \zeta_{\{x\}}(l_0) = \begin{cases} 0.85786\cdots & \text{if } x = 14, \\ 0.14213\cdots & \text{if } x = 15, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2} $$

The measurement $\mathscr{M}(\mathscr{O}_X, S_\Theta)$ (or, its mathematical representation $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$) is sometimes called a *simultaneous measurement* if $\mathscr{O}_X$ (or, its mathematical representation $\mathbf{O}$) is a quasi-product C^*-observable, i.e., the label set X is considered as the product set $\times_{k\in K} X_k (= \{(x_1, x_2, \ldots, x_{|K|}) \colon x_k \in X_k, k \in K \equiv \{1, 2, \ldots, |K|\}\})$. For simplicity, consider a quasi-product C^*-observable $\mathbf{O}_{12} \equiv (X_1 \times X_2, \mathscr{P}_0(X_1 \times X_2), F)$ in a C^*-algebra $\mathscr{A}$. Put $F_1(\Xi_1) = F(\Xi_1 \times X_2)(\forall \Xi_1 \in \mathscr{P}_0(X_1))$ and $F_2(\Xi_2) = F(X_1 \times \Xi_2)(\forall \Xi_2 \in \mathscr{P}_0(X_2))$. For each $k = 1, 2$, the C^*-observable $\mathbf{O}_k \equiv (X_k, \mathscr{P}_0(X_k), F_k)$ in $\mathscr{A}$ is called the *kth marginal C^*-observable* of $\mathbf{O}_{12}$. Also, we sometimes denote $\mathbf{O}_{12}$ by $\mathbf{O}_1 \overset{\mathbf{O}_{12}}{\times} \mathbf{O}_2$, or $(X_1 \times X_2, \mathscr{P}_0(X_1 \times X_2), F_1 \overset{\mathbf{O}_{12}}{\times} F_2)$, that is, $F = F_1 \overset{\mathbf{O}_{12}}{\times} F_2$. Similarly, the simultaneous measurement $\mathbf{M}_{\mathscr{A}}(\mathbf{O}_{12}, S_{\rho^{\mathrm{p}}})$ $(\equiv \mathbf{M}_{\mathscr{A}}(\mathbf{O}_1 \overset{\mathbf{O}_{12}}{\times} \mathbf{O}_2, S_{\rho^{\mathrm{p}}}))$ is sometimes denoted by $\overset{\mathbf{O}_{12}}{\times}_{k=1,2} \mathbf{M}_{\mathscr{A}}(\mathbf{O}_k, S_{\rho^{\mathrm{p}}})$.

Remark 1.12 (*Quasi-product observable in commutative fuzzy theory*). Let $\mathbf{O}_k \equiv (X_k, \mathscr{P}_0(X_k), f^k_{(\cdot)}), k = 1, 2$, be C^*-observables in a commutative C^*-algebra $C_0(\Omega)$. Then, the quasi-product observable $\mathbf{O}_{12}$ $(\equiv \mathbf{O}_1 \overset{\mathbf{O}_{12}}{\times} \mathbf{O}_2) = (X_1 \times X_2, \mathscr{P}_0(X_1 \times X_2), f^1 \overset{\mathbf{O}_{12}}{\times} f^2)$ with the marginal observables $\mathbf{O}_1$ and $\mathbf{O}_2$ always exists. For example, it suffices to put $(f^1 \overset{\mathbf{O}_{12}}{\times} f^2)_{\Xi_1 \times \Xi_2}(\omega) = f^1_{\Xi_1}(\omega) \cdot f^2_{\Xi_2}(\omega)$. Though the uniqueness is not guaranteed in general, the following inequalities hold (cf. [9]):

$$ \max\{0, f^1_{\Xi_1}(\omega) + f^2_{\Xi_2}(\omega) - 1\} \leqslant \left(f^1 \overset{\mathbf{O}_{12}}{\times} f^2 \right)_{\Xi_1 \times \Xi_2} (\omega) \leqslant \min\{f^1_{\Xi_1}(\omega), f^2_{\Xi_2}(\omega)\} $$
$$ (\forall \Xi_k \in \mathscr{P}_0(X_k), k = 1, 2,\ \forall \omega \in \Omega). \tag{1.3} $$

This implies that, if at least one of $\mathbf{O}_1$ and $\mathbf{O}_2$ is crisp, the quasi-product observable $\mathbf{O}_{12}$ is uniquely determined. (This will be again studied in the general situation; cf. Theorem 4.11.) When the uniqueness is guaranteed, the quasi-product observable $\mathbf{O}_1 \overset{\mathbf{O}_{12}}{\times} \mathbf{O}_2$ is sometimes denoted by $\mathbf{O}_1 \overset{\mathrm{u}}{\times} \mathbf{O}_2$.

Consider a fuzzy system S formulated in a C^*-algebra $\mathscr{A}_1$. We sometimes hope to investigate the fuzzy system S in another C^*-algebra $\mathscr{A}_2$. Let $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ be a C^*-homomorphism (i.e., Φ is a continuous linear map such that $\Phi(T_1T_2) = \Phi(T_1)\Phi(T_2)$ and $\Phi(T^*) = \Phi(T)^*$). Note that Φ is also regarded as the C^*-homomorphism from $\bar{\mathscr{A}}_1$ into $\bar{\mathscr{A}}_2$. And let $\Phi^* : \mathscr{A}_2^* \to \mathscr{A}_1^*$ (and so, $\Phi^* : \bar{\mathscr{A}}_2^* \to \bar{\mathscr{A}}_1^*$) be the dual operator of Φ. Let $\mathbf{O}_1 \equiv (X, \mathscr{P}_0(X), F_1)$ be a C^*-observable in $\mathscr{A}_1$ and let $\rho_2^{\mathrm{p}} \in \mathfrak{S}^{\mathrm{p}}(\mathscr{A}_2^*)$. Then we easily see that $\Phi\mathbf{O}_1 \equiv (X, \mathscr{P}_0(X), \Phi F_1)$ be a C^*-observable in $\mathscr{A}_2$ and $\Phi^*\rho_2^{\mathrm{p}} \in \mathfrak{S}^{\mathrm{p}}(\mathscr{A}_1^*)$. Also it is clear that ${}_{\bar{\mathscr{A}}_1^*}\langle \Phi^*\rho_2^{\mathrm{p}}, F_1(\Xi)\rangle_{\bar{\mathscr{A}}_1} = {}_{\bar{\mathscr{A}}_2^*}\langle \rho_2^{\mathrm{p}}, \Phi F_1(\Xi)\rangle_{\bar{\mathscr{A}}_2}$ for all $\Xi \in \mathscr{P}_0(X)$. Therefore, we can assume the following identification:

$$\mathbf{M}_{\mathscr{A}_1}(\mathbf{O}_1 \equiv (X, \mathscr{P}_0(X), F_1), S_{\Phi^*\rho_2^{\mathrm{p}}}) = \mathbf{M}_{\mathscr{A}_2}(\Phi\mathbf{O}_1 \equiv (X, \mathscr{P}_0(X), \Phi F_1), S_{\rho_2^{\mathrm{p}}}). \tag{1.4}$$

Remark 1.13. A fuzzy system S always has its state ρ^{p} ($\in \mathfrak{S}^{\mathrm{p}}(\mathscr{A}^*)$). Thus, it should be denoted by $S_{\rho^{\mathrm{p}}}$. However, in some cases we do not know the state ρ^{p} of the fuzzy system S. For example, if we know the pure state δ_{l_0} in Example 1.11, we may not need to measure the length of the pencil. Hence, we sometimes denote $S_{\rho^{\mathrm{p}}}$ [resp. $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$] by S or $S_{(*)}$ [resp. $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S)$ or $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{(*)})$]. Under the assumption that we do not know the pure state ρ^{p}, $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{(*)})$ is sometimes identified with $\mathbf{O}$.

Remark 1.14. In [9], we proposed the identification: "measurement" = "inference". And, as a consequence of Axioms 0 and 1, we proposed Axioms 1′ (simultaneous measurement) and Axiom 1″ (inference). Though these are quite important, we omit them. We will state them (Methods 2′ and 2″) in the W^*-algebraic formulation in Section 4.

2. Measurement of a frequency probability

The meaning of "probability" in Axiom 1 seems to be a matter of common knowledge in quantum mechanics. However, we, for completeness, discuss about this "probability", i.e., its objective (or, individualistic) aspect. The arguments in this section will also be useful for the investigation in the next section, Section 3.

For each $k = 1, 2, \ldots, n$, consider a measurement $\mathbf{M}_{\mathscr{A}_k}(\mathbf{O}_k \equiv (X, \mathscr{P}(X), F_k), S_{\rho_k^{\mathrm{p}}})$ in a C^*-algebra $\mathscr{A}_k$, where we assume, for simplicity, that X is finite (so $\mathscr{P}_0(X) = \mathscr{P}(X)$). Put $\hat{\mathscr{A}} = \bigotimes_{k=1}^n \mathscr{A}_k$, i.e., the tensor product C^*-algebra of $\{\mathscr{A}_k\colon k = 1, 2, \ldots, n\}$. And so, $\hat{\mathscr{A}}^* = \bigotimes_{k=1}^n \mathscr{A}_k^*$. Though the general theory of tensor product C^*-algebras $\bigotimes_{k=1}^n \mathscr{A}_k$ is not easy (cf. [15]), we only use the following properties (i)–(iii): (i) $T_1 \otimes T_2 \otimes \cdots \otimes T_n \in \hat{\mathscr{A}}$ for any $T_k \in \mathscr{A}_k$, $k = 1, 2, \ldots, n$, (ii) $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n \in \mathfrak{S}^{\mathrm{p}}(\hat{\mathscr{A}}^*)$ for any $\rho_k \in \mathfrak{S}^{\mathrm{p}}(\mathscr{A}_k^*)$, $k = 1, 2, \ldots, n$, (iii) $(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n)(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \prod_{k=1}^n \rho_k(T_k)$ for any $\rho_k \in \mathscr{A}_k^*$ and any $T_k \in \mathscr{A}_k$, $k = 1, 2, \ldots, n$. (If the reader concerns himself to only commutative cases, it is sufficient to know the fact that $\bigotimes_{k=1}^n C_0(\Omega_k) = C_0(\times_{k=1}^n \Omega_k)$ and $\bigotimes_{k=1}^n \mathscr{M}(\Omega_k) = \mathscr{M}(\times_{k=1}^n \Omega_k)$. Therefore, for example, the above (iii) implies the elementary property of product measure (Fubini's theorem), i.e., $\int_{\Omega_1 \times \Omega_2} f_1(\omega_1) \cdot f_2(\omega_2)(\rho_1 \otimes \rho_2)(d\omega_1 d\omega_2) = \int_{\Omega_1} f_1(\omega_1)\rho_1(d\omega_1) \cdot \int_{\Omega_2} f_2(\omega_2)\rho_2(d\omega_2)$.) Here, consider the tensor-product C^*-observable $\bigotimes_{k=1}^n \mathbf{O}_k \equiv (X^n, \mathscr{P}(X^n), \hat{F} \equiv \bigotimes_{k=1}^n F_k)$ in $\hat{\mathscr{A}} (\equiv \bigotimes_{k=1}^n \mathscr{A}_k)$ such that

$$\hat{F}(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n) = F_1(\Xi_1) \otimes F_2(\Xi_2) \otimes \cdots \otimes F_n(\Xi_n) \quad (\forall \Xi_k \in \mathscr{P}(X),\ k = 1, 2, \ldots, n).$$

Therefore, we get the measurement $\mathbf{M}_{\otimes \mathscr{A}_k}(\bigotimes_{k=1}^n \mathbf{O}_k, S_{\otimes_{k=1}^n \rho_k^{\mathrm{p}}})$ in $\bigotimes_{k=1}^n \mathscr{A}_k$, which is also denoted by $\bigotimes_{k=1}^n \mathbf{M}_{\mathscr{A}_k}(\mathbf{O}_k, S_{\rho_k^{\mathrm{p}}})$ and called the *repeated measurement* (or, "*parallel measurement*") of $\mathbf{M}_{\mathscr{A}_k}(\mathbf{O}_k, S_{\rho_k^{\mathrm{p}}})$'s. Put $\mathscr{M}_{+1}(X) = \{\nu\colon \nu$ is a probability measure on X, i.e., $\nu(X) = 1\}$ and define the map $g : X^n \to \mathscr{M}_{+1}(X)$ such that

$$[g(x_1, x_2, \ldots, x_n)](\Xi) = \frac{\sharp[\{k\colon x_k \in \Xi\}]}{n} \quad (\forall \Xi \in \mathscr{P}(X)), \tag{2.1}$$

where $\sharp[B]$ = "the number of the elements of a set B". Then we have the C^*-observable $(\mathscr{M}_{+1}(X), \mathscr{P}_0(\mathscr{M}_{+1}(X)), \hat{F}(g^{-1}(\cdot)))$ in $\hat{\mathscr{A}}$ as the image observable of g for $\bigotimes_{k=1}^n \mathbf{O}_k$, which is called a *frequency* *probability observable*. Though $\mathscr{M}_{+1}(X)$ is not countable, this is not essential (cf. Remark 1.6). That is because the observable can be replaced by $(\mathscr{M}'_{+1}(X), \mathscr{P}(\mathscr{M}'_{+1}(X)), \hat{F}(g^{-1}(\cdot)))$. Here the set $\mathscr{M}'_{+1}(X) \equiv g(X^n)(\subset \mathscr{M}_{+1}(X))$ is clearly finite.

Now we have the following proposition, by which we can show Theorems 2.2 and 3.6.

Proposition 2.1 (The "fuzzy theoretical" weak law of large numbers). *Suppose the above notations. For any* $\varepsilon > 0$ *and any* Ξ $(\in \mathscr{P}(X))$, *define* $\hat{\Xi}_{\Xi,\varepsilon}$ $(\in \mathscr{P}(X^n))$ *by*

$$\hat{\Xi}_{\Xi,\varepsilon} = \left\{ \hat{x} = (x_1, x_2, \ldots, x_n) \in X^n \colon \left| [g(\hat{x})](\Xi) - \frac{1}{n}\sum_{k=1}^{n} \rho_k^{\mathrm{p}}(F_k(\Xi)) \right| < \varepsilon \right\}.$$

Then we see that

$$1 - \frac{1}{4\varepsilon^2 n} \leqslant \left(\bigotimes_{k=1}^{n} \rho_k^{\mathrm{p}} \right) (\hat{F}(\hat{\Xi}_{\Xi,\varepsilon})) \leqslant 1, \quad (\forall \Xi \in \mathscr{P}(X), \forall \varepsilon > 0, \forall n). \tag{2.2}$$

Proof. We easily see that $[g(\hat{x})](\Xi) = \frac{1}{n}\sum_{k=1}^{n} \chi_{\Xi}(I_k(\hat{x}))(\forall \hat{x} = (x_1, x_2, \ldots, x_n) \in X^n)$, where $I_k : X^n \to X$ is defined by $I_k(\hat{x}) \equiv I_k(x_1, x_2, \ldots, x_k, \ldots, x_n) = x_k$ and $\chi_{\Xi} : X \to \mathbf{R}$ is the characteristic function of Ξ (i.e., $\chi_{\Xi}(x) = 1$ $(x \in \Xi), = 0$ $(x \notin \Xi)$). Note that $\chi_{\Xi}(I_k(\cdot)), k = 1, 2, \ldots, n$, are independent variables on a probability measure space $(X^n, \mathscr{P}(X^n), \hat{P}(\cdot) \equiv (\otimes_{k=1}^{n} \rho_k^{\mathrm{p}})(\hat{F}(\cdot)))$. Also it is clear that $\int_{X^n} \chi_{\Xi}(I_k(\hat{x}))\hat{P}(\mathrm{d}\hat{x}) = \int_{X^n} [\chi_{\Xi}(I_k(\hat{x}))]^2 \hat{P}(\mathrm{d}\hat{x}) = \rho_k^{\mathrm{p}}(F_k(\Xi))(k = 1, 2, \ldots, n)$. Therefore, by Čebyšev inequality, we see that

$$\begin{aligned}
\hat{P}(X^n \setminus \hat{\Xi}_{\Xi,\varepsilon}) &= \hat{P}\left(\left\{ \hat{x} \in X^n \colon \left| \frac{\sum_{k=1}^{n} \chi_{\Xi}(I_k(\hat{x}))}{n} - \frac{\sum_{k=1}^{n} \rho_k^{\mathrm{p}}(F_k(\Xi))}{n} \right| \geqslant \varepsilon \right\} \right) \\
&\leqslant \frac{1}{\varepsilon^2 n^2} \int_{X^n} \left| \sum_{k=1}^{n} (\chi_{\Xi}(I_k(\hat{x})) - \rho_k^{\mathrm{p}}(F_k(\Xi))) \right|^2 \hat{P}(\mathrm{d}\hat{x}) \\
&= \frac{1}{\varepsilon^2 n^2} \sum_{k=1}^{n} \int_{X^n} |\chi_{\Xi}(I_k(\hat{x})) - \rho_k^{\mathrm{p}}(F_k(\Xi))|^2 \hat{P}(\mathrm{d}\hat{x}) \\
&\leqslant \frac{1}{\varepsilon^2 n} \max_{1 \leqslant k \leqslant n} \left[\rho_k^{\mathrm{p}}(F_k(\Xi))(1 - \rho_k^{\mathrm{p}}(F_k(\Xi))) \right] \leqslant \frac{1}{4\varepsilon^2 n},
\end{aligned}$$

which implies (2.2). This completes the proof. □

Now we can show the following theorem as an immediate consequence of Proposition 2.1. It clarifies the "probability" in Axiom 1 from the statistical point of view.

Theorem 2.2 (Frequency probability). *Put* $\mathscr{A}_k = \mathscr{A}, \rho_k^{\mathrm{p}} = \rho^{\mathrm{p}}$ *and* $\mathbf{O}_k = \mathbf{O} \equiv (X, \mathscr{P}(X), F), k = 1, 2, \ldots, n$, *in Proposition* 2.1. *Consider the repeated measurement* $\mathbf{M}_{\otimes\mathscr{A}}(\bigotimes_{k=1}^{n} \mathbf{O}, S_{\otimes_{k=1}^{n}\rho^{\mathrm{p}}})$ *in* $\bigotimes_{k=1}^{n} \mathscr{A}$. *Then, we see that*

$$1 - \frac{1}{4\varepsilon^2 n} \leqslant \left(\bigotimes_{k=1}^{n} \rho^{\mathrm{p}} \right) \left(\left(\bigotimes_{k=1}^{n} F \right) \left(\left\{ \hat{x} \in X^n \colon |\rho^{\mathrm{p}}(F(\Xi)) - \frac{\sharp[\{k : x_k \in \Xi\}]}{n}| < \varepsilon \right\} \right) \right) \leqslant 1,$$

$$(\forall \Xi \in \mathscr{P}(X), \forall \varepsilon > 0, \forall n). \tag{2.3}$$

Here note, by Axiom 1, *that* $(\bigotimes_{k=1}^{n} \rho^{\mathrm{p}})((\bigotimes_{k=1}^{n} F)(\hat{\Xi}))$ *is the probability that a measured value by* $\mathbf{M}_{\otimes\mathscr{A}}(\bigotimes_{k=1}^{n} \mathbf{O}, S_{\otimes_{k=1}^{n}\rho^{\mathrm{p}}})$ *belongs to* $\hat{\Xi}$. *Therefore, if n is sufficiently large, for a measured value* $\hat{x}(= (x_1, x_2, \ldots, x_n) \in X^n)$ *by* $\mathbf{M}_{\otimes\mathscr{A}}(\bigotimes_{k=1}^{n} \mathbf{O}, S_{\otimes_{k=1}^{n}\rho^{\mathrm{p}}})$, *we can consider (in the sense of* (2.3)) *that*

$$\rho^{\mathrm{p}}(F(\Xi)) \approx \frac{\sharp[\{k \colon x_k \in \Xi\}]}{n}. \tag{2.4}$$

Remark 2.3. This theorem (i.e., the above (2.4)) connects "individualistic probability" (i.e., "probability in Axiom 1") with "frequency probability". Though some may hope to start from "frequency probability" and not "individualistic probability", it may be impossible. However, we can replace the statement $(*)$ in Axiom 1 by the following $(**)$:

$(**)$ if $\rho^{\mathrm{p}}(F(\Xi)) \approx 1$, we can "almost surely" believe that $x \in \Xi$.

(Here note that the concept "almost surely" belongs to the "individualistic"-category.) Under the condition $(**)$, we can also show Proposition 2.1 (and so Theorem 2.2), and therefore, by (2.4), we can characterize "$\rho^{\mathrm{p}}(F(\Xi))$" as "frequency probability". That is, the statement $(*)$ in Axiom 1 and the above $(**)$ are equivalent.

3. Subjective fuzzy measurement theory

In the previous section, we investigated the objective aspect of fuzzy measurement theory. Under these preparations, in this and the next sections we propose "subjective fuzzy measurement theory". We think that the relation between "objectivity" and "subjectivity" is not simple in general. If we have no "objective theory", we must provide a lot of philosophical arguments for "subjectivity". However, since we already have it, our opinion for "subjectivity" is quite simple. That is, "subjective fuzzy measurement theory" is characterized as "measure theoretical (i.e., statistical) method of (objective) fuzzy measurement theory". (Note that a commutative C^*-algebra $C_0(\Omega)$ belongs to the category of "topology" in mathematics.) Therefore, it may also be called "statistical fuzzy measurement theory". We construct this theory such as it always and naturally has an objective interpretation. Therefore, "subjective fuzzy measurement theory" (proposed in this and the next sections) is never "purely" subjective. Also, it is quite proper to consider that the relation between "objective fuzzy measurement theory" and "subjective fuzzy measurement theory" corresponds to the relation between "Newtonian mechanics" and "statistical mechanics".

Remark 3.1. In this paper we frequently use the ambiguous words such as "objective", "subjective", etc. In most cases, we use them under the identification: "objective" $\approx$ "real" $\approx$ "individualistic", and "subjective" $\approx$ "imaginary" $\approx$ "statistical". As it will be seen throughout this paper (particularly, see Remark 6.5), we believe that this identification is rather pertinent.

We first introduce "imaginary probability" (i.e., "subjective probability"), which is different from "(objective) probability" in Axiom 1.

Definition 3.2 (*Imaginary C^*-measurement*). Let $\mathscr{A}$ be a C^*-algebra. Let $\mathbf{O} \equiv (X, \mathscr{P}_0(X), F)$ be a C^*-observable in $\mathscr{A}$ and let $\rho^{\mathrm{m}} \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$. Then, the mixed state ρ^{m} is also called an imaginary state (or, subjective state). Also, the symbol $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S(\rho^{\mathrm{m}}))$ is called an imaginary C^*-measurement (or, subjective C^*-measurement) in $\mathscr{A}$.

Note that "imaginary C^*-measurement" is defined in mathematics. Thus, it has no reality in itself since the state of a system S is always represented by a pure state ρ^{p} and not a mixed state ρ^{m} (cf. Axiom 0). Therefore, the following "method" (i.e., "subjective fuzzy measurement theory") should be read in mathematics. That is because we have no experiment that tests the statement in Method 1 directly (cf. Theorem 3.6).

Method 1. *Consider an imaginary C^*-measurement $\mathbf{M}_{\mathscr{A}}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), F), S(\rho^{\mathrm{m}}))$ in a C^*-algebra $\mathscr{A}$. Then, we consider that*

$(*)$ *the "imaginary probability" (or, "subjective probability") that x $(\in X)$, the measured value by the imaginary C^*-measurement $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S(\rho^{\mathrm{m}}))$, belongs to a set $\Xi (\in \mathscr{P}_0(X))$ is given by $\rho^{\mathrm{m}}(F(\Xi)) (\equiv_{\mathscr{A}^*} \langle \rho^{\mathrm{m}}, F(\Xi) \rangle_{\mathscr{A}})$.*

As stated later (in Section 4), "subjective fuzzy measurement theory" is also described in a W^*-algebraic formulation (cf. Method 2 in Section 4). In order to bridge the gap between Axiom 1 and Method 2, Method 1 is useful. However, we do not think that Method 1 is temporary. It is quite important in itself.

Remark 3.3 (*Kolmogorov's method*). Again note that Method 1 is meaningless from the objective point of view. Here recall Kolmogorov's probability theory (cf. [11]). He did not propose a principle (such as Axioms 0 and 1) but only gave the mathematical definition of the probability space $(X, \mathscr{F}, P)$. And furthermore, he introduced the "method" (i.e., Kolmogorov's method) such as

(∗∗) "the probability that an event Ξ ($\in \mathscr{F}$) occurs is given by $P(\Xi)$".

In spite of its utility, his method is also clearly meaningless from the objective point of view. Though it seems to be somewhat strange that a meaningless statement is quite important, this is a fact. (The reason will become clear throughout this paper; cf. Remark 6.5.)

As general arguments, we first mention two main advantages of Method 1 in the following Remarks 3.4 and 3.5.

Remark 3.4 (*Preciseness*). Note that one of the most important (subjective) interpretation of Kolmogorov's method (∗∗) is obtained by putting $(X, \mathscr{F}, P(\cdot)) = (X, \bar{\mathscr{P}}_0(X), \rho^{\mathrm{m}}(F(\cdot)))$ in Method 1 (cf. Remark 1.6 or Definition 4.5). Therefore, we can find "Kolmogorov's theory" everywhere in fuzzy measurement theory. Though these two methods (i.e., Method 1 and Kolmogorov's method) are meaningless, we may say that Method 1 is more "precise" than Kolmogorov's method. That is, Method 1 is near Axiom 1. Therefore, it is natural to expect that Method 1 will produce rather precise results. For example, we can directly see "membership functions" (="fuzzy sets") in Method 1. Therefore, we can propose useful methods concerning membership functions (cf. Sections 5 and 6).

Remark 3.5 (*Objectivity*). In general, the mathematical concept has many interpretations. Therefore, if we want to assert an objective statement by using "method" (i.e., "mathematics"), we must always add an objective interpretation to "mathematics". In fact, Kolmogorov said himself that his theory should be used like this. However, this is impossible in the strict sense. That is because he did not teach us "What is objective?" (that is, he did not propose "objective theory"). On the other hand, we can state the objective interpretation of Method 1 within Axioms 0 and 1 (cf. Theorem 3.6 and Remark 4.8).

After all, all advantages are due to the fact that we already have (objective) fuzzy measurement theory. Of course, we can state many other advantages, for example, Method 1 is applicable to quantum mechanics. However, we mainly concern ourselves to the above two advantages.

First, we study the advantage concerning "objectivity" (cf. Theorem 3.6 and Remark 4.8). The reader will find the advantage concerning "preciseness" everywhere in Sections 5 and 6. Let $\mathbf{M}_{\mathscr{A}}(\mathbf{O} \equiv (X, \mathscr{P}(X), F), S(\rho^{\mathrm{m}}))$ be an imaginary C^*-measurement in a C^*-algebra $\mathscr{A}$ where X is, for simplicity, assumed to be finite. Let $\{\rho_k^{\mathrm{p}}\}_{k=1}^{\infty}$ be a sequence in $\mathfrak{S}^{\mathrm{p}}(\mathscr{A}^*)$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \rho_k^{\mathrm{p}} \to \rho^{\mathrm{m}} \quad \text{as } n \to \infty \qquad (\text{in the weak}^* \text{ topology } \sigma(\mathscr{A}^*; \mathscr{A})). \tag{3.1}$$

(The existence of the $\{\rho_k^{\mathrm{p}}\}_{k=1}^{\infty}$ is guaranteed by Krein–Milman theorem (cf. [17]). Let ε be any positive real number. The (3.1) implies that there exists a natural number N ($\equiv N(\varepsilon, \mathbf{O})$) such that

$$\left| \frac{1}{n}\sum_{k=1}^{n} \rho_k^{\mathrm{p}}(F(\Xi)) - \rho^{\mathrm{m}}(F(\Xi)) \right| < \varepsilon/2 \qquad (\forall \Xi \in \mathscr{P}(X),\ \forall n \geqslant N). \tag{3.2}$$

Consider the repeated measurement $\mathbf{M}_{\otimes\mathscr{A}}((X^n,\mathscr{P}(X^n),\bigotimes_{k=1}^n F),S_{\otimes_{k=1}^n \rho_k^{\rm p}})$ in $\bigotimes_{k=1}^n \mathscr{A}$. For any $\Xi(\in \mathscr{P}(X))$, define $\hat{\Xi}_{\Xi,\varepsilon}(\in \mathscr{P}(X^n))$ by

$$\hat{\Xi}_{\Xi,\varepsilon} = \{\hat{x} = (x_1,x_2,\dots,x_n) \in X^n\colon\ |[g(\hat{x})](\Xi) - \rho^{\rm m}(F(\Xi))| < \varepsilon\}, \tag{3.3}$$

where $g\colon X^n \to \mathscr{M}_{+1}(X)$ is defined as in (2.1). Assume that $n \geqslant N$. Then we see, by (3.2) and (3.3), that

$$\hat{\Xi}_{\Xi,\varepsilon} \supseteq \left\{\hat{x} \in X^n\colon \left|[g(\hat{x})](\Xi) - \frac{1}{n}\sum_{k=1}^{n} \rho_k^{\rm p}(F(\Xi))\right| < \varepsilon/2\right\} (\equiv \hat{\Xi}^0_{\Xi,\varepsilon}).$$

Here note, by Proposition 2.1, that

$$1 - \frac{1}{\varepsilon^2 n} \leqslant \left(\bigotimes_{k=1}^{n} \rho_k^{\rm p}\right)\left(\left(\bigotimes_{k=1}^{n} F\right)(\hat{\Xi}^0_{\Xi,\varepsilon})\right) \leqslant 1.$$

Thus, we get the following theorem.

Theorem 3.6 (An objective interpretation of "imaginary C^*-measurement"). *Let $\mathbf{M}_{\mathscr{A}}(\mathbf{O} \equiv (X, \mathscr{P}(X), F), S(\rho^{\rm m}))$ be an imaginary C^*-measurement in a C^*-algebra $\mathscr{A}$. Let $\{\rho_k^{\rm p}\}_{k=1}^{\infty}$ be a sequence in $\mathfrak{S}^{\rm p}(\mathscr{A}^*)$ that satisfies* (3.1). *Let N be such as* (3.2). *Then we see that*

$$1 - \frac{1}{\varepsilon^2 n} \leqslant \left(\bigotimes_{k=1}^{n} \rho_k^{\rm p}\right)\left(\left(\bigotimes_{k=1}^{n} F\right)\left(\left\{\hat{x} \in X^n\colon \left|\rho^{\rm m}(F(\Xi)) - \frac{\sharp[\{k\colon x_k \in \Xi\}]}{n}\right| < \varepsilon\right\}\right)\right) \leqslant 1,$$

$$(\forall \Xi \in \mathscr{P}(X),\ \forall \varepsilon > 0,\ \forall n \geqslant N). \tag{3.4}$$

Therefore, if n is sufficiently large, for a measured value $\hat{x}(= (x_1,x_2,\dots,x_n) \in X^n)$ of the repeated measurement $\mathbf{M}_{\otimes\mathscr{A}}(\bigotimes_{k=1}^n \mathbf{O},\ S_{\otimes_{k=1}^n \rho_k^{\rm p}})$, we can consider (in the sense of (3.4)) *that*

$$\rho^{\rm m}(F(\Xi)) \approx \frac{\sharp[\{k\colon x_k \in \Xi\}]}{n}. \tag{3.5}$$

That is, in the sense of (3.4) *or* (3.5), *the imaginary C^*-measurement $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S(\rho^{\rm m}))$ is realized by the repeated measurement $\mathbf{M}_{\otimes\mathscr{A}}(\bigotimes_{k=1}^n \mathbf{O},\ S_{\otimes_{k=1}^n \rho_k^{\rm p}})$.*

This theorem is most fundamental in "subjective fuzzy measurement theory". That is because it guarantees that we can always and naturally add an objective interpretation to Method 1. In other words, the above theorem instructs us how to read Method 1 from the objective point of view (cf. Remark 3.5). For a fixed $\{\rho_k^{\rm p}\}_{k=1}^{\infty}$, we can conduct an experiment that tests Method 1.

Example 3.7 (*The principle of equal weigh in statistical mechanics*). Let Ω be a compact Hausdorff space. And consider a commutative C^*-algebra $\mathscr{A} \equiv C(\Omega)$. Let $\phi\colon \Omega \to \Omega$ be a bicontinuous map. And furthermore, assume that ϕ is unique ergodic. That is, there exists a unique probability measure ν on Ω such that $\nu(D) = \nu(\phi(D))$ for any Borel set D in Ω. Here the ν is sometimes called the equilibrium state. Then we see, by the ergodic theorem concerning Markov operator (cf. [12]), that, for any ω_0 $(\in \Omega)$,

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{\phi^k\omega_0} \to \nu \quad \text{as } n \to \infty \qquad (\text{in the weak}^* \text{ topology } \sigma(\mathscr{M}(\Omega); C(\Omega))),$$

where $\phi^k\omega_0 = \phi(\phi^{k-1}\omega_0)$. Therefore, when Ω is a compact subset of the phase space $\mathbf{R}^{6N}$ and $\{\phi^k\colon k = 0,\pm 1,\pm 2,\dots\}$ is a unique ergodic dynamical flow on Ω, Theorem 3.6 gives an objective interpretation to "the principle of equal weight" in statistical mechanics. That is, for any C^*-observable $\mathbf{O}$, the imaginary

C^*-measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S(v))$ is realized by the repeated measurement $\mathbf{M}_{\otimes C(\Omega)}(\bigotimes_{k=1}^{n} \mathbf{O}, S_{\otimes_{k=1}^{n} \delta_{\phi^k \omega_0}})$ for sufficiently large n.

Remark 3.8 (*Classical mechanics*). As the background of Example 3.7, we of course assume that a classical mechanical system is a kind of fuzzy system, that is, we view "classical mechanics" (i.e., "Newtonian mechanics") as

$$\text{"classical mechanics"} = \text{"commutative fuzzy theory"} + \text{"Newtonian equation"}. \tag{3.6}$$

This is natural since "measurement" is most basic in every science. Here note that the standard syllogism (shown in [9]) is frequently used (as an obvious fact) in classical mechanics. Also, this view-point (3.6) is a matter of common knowledge in quantum mechanics, that is, "quantum mechanics" = "Born's axiom (i.e., non-commutative fuzzy theory)" + "Heisenberg's kinetic equation".

Under the hypothesis (3.6), we can explain Example 1.11 (the measurement of pencil's length) from the physical point of view.

Remark 3.9 (*Physical explanation for the measurement of pencil's length*). Assume that the pencil (in Example 1.11) is composed of N-particles. Let $\Omega = \mathbf{R}^{6N}$ be a phase space, whose point represents the state of N-particles system. Therefore, we consider that the pencil is represented by a certain point ω_0 $(\in \Omega)$. Define the continuos map ϕ: $\Omega \to \mathbf{R}$ such that $\phi(\omega)$="the actual length (or, diameter) of the N-particles system represented by a phase space point ω". Thus we see that $l_0 = \phi(\omega_0)$. And define the C^*-homomorphism Φ: $\overline{C_0(\mathbf{R})} \to \overline{C_0(\Omega)}$ such that $\overline{C_0(\mathbf{R})} \ni f(\cdot) \overset{\Phi}{\mapsto} f(\phi(\cdot)) \in \overline{C_0(\Omega)}$. Let $\mathbf{O}_{\mathscr{Z}} = (\mathbb{Z}, \mathscr{P}_0(\mathbb{Z}), \zeta_{(\cdot)})$ be the fuzzy numbers observable in $C_0(\mathbf{R})$ as in Example 1.11. Here we see, by (1.4) and $\Phi^*\delta_{\omega_0} = \delta_{l_0}$, that

$$\mathbf{M}_{C_0(\boldsymbol{R}^{6N})}(\Phi\mathbf{O}_{\mathscr{Z}}, S_{\delta_{\omega_0}}) = \mathbf{M}_{C_0(\boldsymbol{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}}).$$

Note that the $\mathbf{M}_{C_0(\boldsymbol{R}^{6N})}(\Phi\mathbf{O}_{\mathscr{Z}}, S_{\delta_{\omega_0}})$ has the reality under the hypothesis (3.6). Therefore, the measurement $\mathbf{M}_{C_0(\boldsymbol{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}})$ in Example 1.11 also has the reality.

There seems to be an opinion that an imaginary C^*-measurement does not always need an objective interpretation. We partially agree to this opinion. That is, if our concerning is not "true or not true" but "useful or not useful", we do not consider that we must explicitly show the objective interpretation of the imaginary C^*-measurement.

Definition 3.10 (*Objective and subjective C^*-measurement*). Let $\mathscr{A}$ be a C^*-algebra. Let $\rho^{\mathrm{p}} \in \mathfrak{S}^{\mathrm{p}}(\mathscr{A}^*)$ and $\rho^{\mathrm{m}} \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$. Then, the symbol $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}}(\rho^{\mathrm{m}}))$ is called a real and imaginary C^*-measurement (or, objective and subjective C^*-measurement) in $\mathscr{A}$. That is, we consider that $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}}(\rho^{\mathrm{m}})) = \mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}})$ from the objective point of view, and $\mathbf{M}_{\mathscr{A}}(\mathbf{O}, S_{\rho^{\mathrm{p}}}(\rho^{\mathrm{m}})) = \mathbf{M}_{\mathscr{A}}(\mathbf{O}, S(\rho^{\mathrm{m}}))$ from the subjective point of view.

Example 3.11 (*The subjective probability in the measurement of pencil's length*). Now we investigate the measurement of pencil's length from the subjective point of view. Let $\mathscr{A} = C_0(\mathbf{R})$, S, $l_0 = 14.1421\ldots$, $\mathbf{O}_{\mathscr{Z}} = (\mathbb{Z}, \mathscr{P}_0(\mathbb{Z}), \zeta_{(\cdot)})$ be as in Example 1.11. Assume that the (subjective) state of the system S is ρ_0 $(\in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*) \equiv \mathscr{M}_{+1}(\mathbf{R}))$. For example, put $\rho_0(D) = \frac{1}{10}\int_{10}^{20} \chi_D(\omega) m(\mathrm{d}\omega)$ $(\forall D \in \mathscr{B}_{\boldsymbol{R}})$ where m is the Lebesgue measure on $\mathbf{R}$. (There is no absolute reason that the ρ_0 is absolute continuous with respect to the Lebesgue measure m.) Probably, the subjective state ρ_0 is determined by a rough eye measurement. Now consider the objective and subjective C^*-measurement $\mathbf{M}_{C_0(\boldsymbol{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}}(\rho_0))$. Note that this is equal to $\mathbf{M}_{C_0(\boldsymbol{R})}(\mathbf{O}_{\mathscr{Z}}, S(\rho_0))$ from the subjective point of view. Then, by Method 1, we consider the following.

(*) the (subjective) probability that the measured value z ($\in \mathbb{Z}$) belongs to $\Xi(\in \mathscr{P}_0(\mathbb{Z}))$ is given by ${}_{\mathscr{M}(R)}\langle \rho_0, \zeta_\Xi\rangle_{\overline{C_0(R)}} = \int_R \zeta_\Xi(\omega)\rho_0(\mathrm{d}\omega)\frac{1}{10}\int_{10}^{20} \zeta_\Xi(\omega)m(\mathrm{d}\omega)$.

Therefore, if $\Xi = \{x\}$, it holds:

$$ {}_{\mathscr{M}(\mathbf{R})}\langle \rho_0, \zeta_\Xi\rangle_{\overline{C_0(\mathbf{R})}} = \begin{cases} \frac{1}{20} & \text{if } x = 10, 20, \\ \frac{1}{10} & \text{if } x = 11, 12, \ldots, 18, 19, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7} $$

Of course, from the objective point of view, we get the same result as (1.2). However, in the above situation, we do not know the pure state δ_{l_0}.

Remark 3.12 (*Fuzzy measure theory*). Since Method 1 is meaningless in itself, we have no absolute reason that the "subjective state" ρ^{m} should be taken in $\mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$. The only one reason is due to Krein–Milman theorem (and so, Theorem 3.6). That is, if we want the "objectivity" (i.e., if we want to test the statement represented by Method 1), we can always and naturally add an objective interpretation to Method 1. Therefore, if our concerning is not "true or not true" but "useful or not useful", we have no reason to consider that $\rho^{\mathrm{m}} \in \mathfrak{S}^{\mathrm{m}}(\mathscr{A}^*)$. As this kind of the generalization, "fuzzy measure" seems to be standard in the case that $\mathscr{A} = C_0(\Omega)$ (cf. [16]). This is "purely" subjective and never statistical. Therefore, "fuzzy measure theory" is the exceptional case of Remark 3.1 (Also, see Remark 6.5.)

4. W^*-algebraic formulation of subjective fuzzy measurement theory

In order to develop "subjective fuzzy measurement theory", in this section we introduce the W^*-algebraic formulation of subjective fuzzy measurement theory.

We have an opinion that a principle (i.e., "objective theory") should be described by a "topological" formulation (i.e., C^*-algebraic formulation). On the other hand, a "measure theoretical" formulation (i.e., W^*-algebraic formulation) is convenient to describe "methods". (Though Method 1 is formulated by C^*-algebras, we have several restrictions such as the label set X is countable.)

Let $\mathscr{N}$ be a W^*-algebra (i.e., von Neumann algebra), that is, $\mathscr{N}$ is a C^*-algebra with the predual Banach space $\mathscr{N}_*$ (i.e., $\mathscr{N} = (\mathscr{N}_*)^*$). Then, we can define the *normal state-class* $\mathfrak{S}^n(\mathscr{N}_*)$ such as

$$ \mathfrak{S}^n(\mathscr{N}_*) \equiv \{\bar{\rho} \in \mathscr{N}_*: \|\bar{\rho}\|_{\mathscr{N}_*} = 1 \text{ and } \bar{\rho} \geqslant 0 \text{ (i.e., } \bar{\rho}(T^*T) \geqslant 0 \text{ for all } T \in \mathscr{N})\}. $$

The element $\bar{\rho}$ of $\mathfrak{S}^n(\mathscr{N}_*)$ is called a *normal state* (or, *density state*). The linear functional $\bar{\rho}(T)$ is sometimes denoted by ${}_{\mathscr{N}_*}\langle \bar{\rho}, T\rangle_{\mathscr{N}}$. Also, note that a W^*-algebra $\mathscr{N}$ has a lot of projections, that is, the set of all finite linear combinations of projections is dense in $\mathscr{N}$ in the weak* topology $\sigma(\mathscr{N}; \mathscr{N}_*)$.

Example 4.1. (i) Let $(\Omega, \mathscr{B}_\Omega, \mu)$ is a measure space. For any $1 \leqslant p \leqslant \infty$, define $L^{\mathrm{p}}(\Omega; \mu) = \{f:\ f$ is a complex valued measurable function such that $\|f\|_{L^{\mathrm{p}}} \equiv [\int_\Omega |f(\omega)|^{\mathrm{p}}\mu(\mathrm{d}\omega)]^{1/\mathrm{p}} < \infty\}$. Then, the $\mathscr{N} \equiv L^\infty(\Omega; \mu)$ is a commutative W^*-algebra with the predual Banach space $\mathscr{N}_* = L^1(\Omega; \mu)$. Of course $\mathfrak{S}^n(\mathscr{N}_*) = L^1_{+1}(\Omega; \mu) \equiv \{\bar{\rho} \in L^1(\Omega; \mu)$: $\bar{\rho} \geqslant 0, \int_\Omega \bar{\rho}(\omega)\mu(\mathrm{d}\omega) = 1$, i.e., $\bar{\rho}$ is a density function$\}$. Also, it is well known that any commutative W^*-algebra $\mathscr{N}$ is represented by some $L^\infty(\Omega; \mu)$.

(ii) When $\mathscr{N} = B(V)$, we see that $\mathscr{N}_* = \mathrm{Tr}(V)$ (cf. Example 1.4) and $\mathfrak{S}^n(\mathscr{N}_*) = \mathrm{Tr}_{+1}(V) \equiv \{\bar{\rho} \in \mathrm{Tr}(V)$: $\bar{\rho} \geqslant 0, \|\bar{\rho}\|_{\mathrm{Tr}(V)} = 1\}$. Also, note that ${}_{\mathrm{Tr}(V)}\langle \bar{\rho}, T\rangle_{B(V)} = \mathrm{tr}[\bar{\rho} \cdot T]_V$, where $\mathrm{tr}[A]_V \equiv \sum_{\lambda\in\Lambda}\langle e_\lambda, Ae_\lambda\rangle_V$. Here, it is well known that the value $\mathrm{tr}[A]_V$ is independent of the choice of a complete orthonormal basis $\{e_\lambda \mid \lambda \in \Lambda\}$ in V. Also, any $\bar{\rho}$ ($\in \mathrm{Tr}_{+1}(V)$) is represented by $\bar{\rho} = \sum_{\lambda\in\lambda}\alpha_\lambda|e_\lambda\rangle\langle e_\lambda|$ (in the trace norm $\|\cdot\|_{\mathrm{Tr}(V)}$) for some complete orthonormal basis $\{e_\lambda|\lambda \in \Lambda\}$ in V and some sequence $\{a_\lambda\}_{\lambda\in\Lambda}$ of non-negative numbers such that $\sum_{\lambda\in\Lambda} a_\lambda = 1$.

The following definition is the W^*-algebraic version of Definition 1.5.

Definition 4.2 (*W^*-observable*). Let $\mathcal{N}$ be a W^*-algebra. A W^*-observable (or, measure theoretical observable) $\bar{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ in $\mathcal{N}$ is defined such that it satisfies that

(i) $(X, \mathcal{F})$ is a measurable space, that is, $\mathcal{F}$ is a σ-field on X,

(ii) for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a positive element in $\mathcal{N}$ (i.e., $0 \leqslant F(\Xi) \in \mathcal{N}$) such that $F(\emptyset) = 0$ and $F(X) = I$, where 0 is the 0-element and I is the identity element in $\mathcal{N}$, and

(iii) for any countable decomposition $\{\Xi_j\}_{j=1}^{\infty}$ of Ξ, $(\Xi_j, \Xi \in \mathcal{F})$, $F(\Xi) = \sum_{j=1}^{\infty} F(\Xi_j)$ holds where the series is convergent in the sense of the weak*-topology $\sigma(\mathcal{N}; \mathcal{N}_*)$ in $\mathcal{N}$.

If $F(\Xi)$ is a projection for every $\Xi(\in \mathcal{F})$, a W^*-observable $(X, \mathcal{F}, F)$ in $\mathcal{N}$ is called a crisp W^*-observable in $\mathcal{N}$.

Example 4.3 (*Crisp W^*-observables*). (i) As a typical crisp W^*-observable in $L^{\infty}(\Omega; \mu)$, the *fundamental observable* $\bar{\mathbf{O}}_{\mathrm{FND}} \equiv (\Omega, \mathcal{B}_{\Omega}, \chi_{(\cdot)})$ is frequently used where χ_{Ξ} is the characteristic function of $\Xi(\in \mathcal{B}_{\Omega})$. This observable is finest in $L^{\infty}(\Omega; \mu)$, i.e., it includes all projections.

(ii) Consider the commutative W^*-algebra $L^{\infty}(\Omega; \mu)$. Let $a\colon \Omega \to \mathbf{R}$ be a measurable function. Then, we can define the crisp W^*-observable $\bar{\mathbf{O}}_a = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, f_{(\cdot)})$ in $L^{\infty}(\Omega, \mu)$ such that $f_{\Xi}(\omega) = \chi_{a^{-1}(\Xi)}(\omega)$ $(\forall \Xi \in \mathcal{B}_{\mathbf{R}}, \forall \omega \in \Omega)$. Note that we can identify the $a(\omega)$ with the $\bar{\mathbf{O}}_a$. That is because $f_{(-\infty,\lambda)}(\omega) = 0$ (if $\lambda < a(\omega)$) $= 1$ (if $\lambda \geqslant a(\omega)$), and therefore, the $a(\omega)$ is determined by the equality $a(\omega) = \int_{\mathbf{R}} \lambda \delta_{a(\omega)}(\mathrm{d}\lambda) = \int_{\mathbf{R}} \lambda f_{\mathrm{d}\lambda}(\omega)$ (a.e. μ).

(iii) Next consider the quantum version of the above (ii). Let A be a self-adjoint operator (not necessarily bounded) on a Hilbert space V. Note that it has the spectral representation: $A = \int_{\mathbf{R}} \lambda E_A(\mathrm{d}\lambda)$. Here, the spectral measure $\bar{\mathbf{O}}_A \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E_A)$ is of course the crisp W^*-observable in $B(V)$. Conversely, any crisp W^*-observable $(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F)$ in $B(V)$ determines a unique self-adjoint operator A_F on V such that $A_F = \int_{\mathbf{R}} \lambda F(\mathrm{d}\lambda)$. Therefore, under the identification: $A \leftrightarrow \bar{\mathbf{O}}_A$, the spectral measure $\bar{\mathbf{O}}_A$ is often denoted by $[A]$.

Remark 4.4. Note that the above examples are all crisp. As mentioned before, a W^*-algebra has a lot of projections, and so, sufficiently many crisp observables. (Also, compared to Remark 1.8.) Thus, in most cases, we can do well without "fuzzy observables" in the W^*-algebraic formulation. This fact sometimes makes us blind to the importance of "fuzzy observables". However, we must not overlook it because the W^*-algebraic formulation is one of methods of (objective) fuzzy measurement theory.

Definition 4.5 (*Imaginary W^*-measurement*). Let $\mathcal{N}$ be a W^*-algebra. Let $\bar{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ be a W^*-observable in $\mathcal{N}$ and let $\bar{\rho} \in \mathfrak{S}^n(\mathcal{N}_*)$. Then, the symbol $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$ is called an imaginary W^*-measurement (or in short, W^*-measurement) in $\mathcal{N}$. Also, the normal state $\bar{\rho}$ is called an imaginary state (or, subjective state). Define the set function $P\colon \mathcal{F} \to \mathbf{R}$ such that $P(\Xi) = \bar{\rho}(F(\Xi))$ $(\forall \Xi \in \mathcal{F})$. Then we can easily see, from the σ-additivity of (iii) in Definition 4.2, that the $(X, \mathcal{F}, P)$ is a probability space.

The following "method" is a W^*-algebraic version of Method 1. Therefore, it also has no reality in itself (cf. Remark 3.3).

Method 2. *Consider a W^*-measurement $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}} \equiv (X, \mathcal{F}, F), S(\bar{\rho}))$ in a W^*-algebra $\mathcal{N}$. Then, we consider that*

$(*)$ *the "imaginary probability" (or,"subjective probability") that $x(\in X)$, the measured value obtained by the W^*-measurement $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$, belongs to a set $\Xi(\in \mathcal{F})$ is given by $\bar{\rho}(F(\Xi))(\equiv_{\mathcal{N}_*} \langle \bar{\rho}, F(\Xi) \rangle_{\mathcal{N}})$.*

Remark 4.6 (*Quantum theory*). Note that Method 2 (as well as Method 1) is meaningless from the objective point of view. However, it should be noted that Born's axiom is usually regarded as Method 2 for $\mathscr{N} = B(V)$. This is due to the peculiarity of quantum mechanics. That is, it holds that $\mathscr{C}(V)^* = \mathrm{Tr}(V) = B(V)_*$. Therefore, for any $\rho^{\mathrm{p}} \in \mathfrak{S}^{\mathrm{p}}(\mathscr{C}(V)^*)(\subseteq \mathfrak{S}^{\mathrm{m}}(\mathscr{C}(V)^*) = \mathfrak{S}^n(B(V)_*))$ and any W^*-observable $(X, \mathscr{F}, F)$ in $B(V)$, we can naturally define the "linear functional" ${}_{C(V)^*}\langle \rho^{\mathrm{p}}, F(\Xi)\rangle_{B(V)}$. Therefore, Method 2 for $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}, S(\rho^{\mathrm{p}}))$ can be regarded as the principle (i.e., Born's axiom) in quantum mechanics. On the other hand, "objectivity" and "subjectivity" are completely separated in classical systems.

Remark 4.7 (*Normalized W^*-measurement in commutative W^*-algebra*). Let $\mathbf{M}_{L^\infty(\Omega;\mu)}(\bar{\mathbf{O}} \equiv (X, \mathscr{F}, \bar{f}_{(\cdot)}), S(\bar{\rho}))$ be a W^*-measurement in a commutative W^*-algebra $L^\infty(\Omega;\mu)$. Define the probability space (i.e., normalized measure space) $(\Omega, \mathscr{B}_\Omega, P)$ such that $P(D) = \int_D \bar{\rho}(\omega)\mu(d\omega)$ $(\forall D \in \mathscr{B}_\Omega)$. Then, we have the *normalized* W^*-measurement $\mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$ in the *normalized* commutative W^*-algebra $L^\infty(\Omega;P)$, where 1 $(\in L^1_{+1}(\Omega;P))$ is the constant function on Ω with its value 1. We of course identify these two W^*-measurements, that is,

$$\mathbf{M}_{L^\infty(\Omega;\mu)}(\bar{\mathbf{O}}, S(\bar{\rho})) = \mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1)). \tag{4.1}$$

Also, it is clear that ${}_{L^1(\Omega;\mu)}\langle \bar{\rho}, \bar{f}_\Xi\rangle_{L^\infty(\Omega;\mu)} = {}_{L^1(\Omega;P)}\langle 1, \bar{f}_\Xi\rangle_{L^\infty(\Omega;P)}$ holds for all Ξ $(\in \mathscr{F})$.

Now we consider the relation between Methods 1 and 2. Though this should be done for general imaginary C^*-measurements in $\mathscr{A}$ (by using GNS-construction, cf. [15]), for simplicity we restrict ourselves to only commutative cases (i.e., $\mathscr{A} = C_0(\Omega)$).

Let $\mathbf{M}_{C_0(\Omega)}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), f_{(\cdot)}), S(\rho^{\mathrm{m}}))$ be an imaginary C^*-measurement in a commutative C^*-algebra $C_0(\Omega)$. Note that the label set X is always assumed to be at most countable. Here consider the commutative W^*-algebra $L^\infty(\Omega;\rho^{\mathrm{m}})$. For any $\Xi(\in \mathscr{P}(X))$, define the membership function $\bar{f}_\Xi$ such that $\bar{f}_\Xi(\omega) = \sum_{x\in\Xi} f_{\{x\}}(\omega)$ $(\forall\omega \in \Omega)$. Note that $\bar{f}_\Xi = f_\Xi$ holds for $\Xi(\in \mathscr{P}_0(X) \subseteq \mathscr{P}(X))$. Then, we get the W^*-observable $\bar{\mathbf{O}} \equiv (X, \mathscr{P}(X), \bar{f}_{(\cdot)})$ in $L^\infty(\Omega;\rho^{\mathrm{m}})$. Consider the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;\rho^{\mathrm{m}})}(\bar{\mathbf{O}} \equiv (X, \mathscr{P}(X), \bar{f}_{(\cdot)}), S(1))$, which is regarded as a *(normalized) W^*-algebraic representation* of $\mathbf{M}_{C_0(\Omega)}(\mathbf{O}, S(\rho^{\mathrm{m}}))$. We also denote that

$$\mathbf{M}_{C_0(\Omega)}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), f_{(\cdot)}), S(\rho^{\mathrm{m}})) \approx_{W^*} \mathbf{M}_{L^\infty(\Omega;\rho^{\mathrm{m}})}(\bar{\mathbf{O}} \equiv (X, \mathscr{P}(X), \bar{f}_{(\cdot)}), S(1)). \tag{4.2}$$

Here note that ${}_{\mathscr{M}(\Omega)}\langle \rho^{\mathrm{m}}, f_\Xi\rangle_{\overline{C_0(\Omega)}} = {}_{L^1(\Omega;\rho^{\mathrm{m}})}\langle 1, \bar{f}_\Xi\rangle_{L^\infty(\Omega;\rho^{\mathrm{m}})}$ for all $\Xi(\in \mathscr{P}_0(X) \subseteq \mathscr{P}(X))$. The representation is not of course unique. For example, consider the measure μ (on Ω) and the normal state $\bar{\rho}(\in L^1_{+1}(\Omega;\mu))$ such that $\rho^{\mathrm{m}}(D) = \int_D \bar{\rho}(\omega)\mu(d\omega)$ $(\forall D \in \mathscr{B}_\Omega)$. Then, by using (4.1) and (4.2), we get the following representation:

$$\mathbf{M}_{C_0(\Omega)}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), f_{(\cdot)}), S(\rho^{\mathrm{m}})) \approx_{W^*} \mathbf{M}_{L^\infty(\Omega;\mu)}(\bar{\mathbf{O}} \equiv (X, \mathscr{P}(X), \bar{f}_{(\cdot)}), S(\bar{\rho})). \tag{4.3}$$

Next consider the reverse of the above argument. Without loss of generality (cf. Remark 4.7), we begin with the normalized W^*- measurement $\mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}} \equiv (X, \mathscr{F}, \bar{f}_{(\cdot)}), S(1))$. Here assume that Ω is a locally compact space, and X is separable metric space. Let Δ be any (small) positive real number. Take a countable decomposition $\{D_1, D_2, \ldots\}$ of X, D_k is a Borel set in X, such that "the diameter of D_k" $< \Delta(\forall k)$. Let N be a natural number such that $\sum_{k=N}^\infty \int_\Omega \bar{f}_{D_k}(\omega)P(d\omega) < \Delta$.Put $\bar{f}'_{\{D_k\}} = \bar{f}_{D_k}$ $(k = 1, 2, \ldots, N-1)$ and $\bar{f}'_{\{D_N\}} = 1 - \sum_{k=1}^{N-1} \bar{f}_{D_k}$. Then, we get the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}_\Delta \equiv (X' = \{D_1, D_2, \ldots, D_N\}, \mathscr{P}(X'), \bar{f}'_{(\cdot)}), S(1))$. From this, we can also get the imaginary C^*-measurement $\mathbf{M}_{C_0(\Omega)}(\mathbf{O}^c_\Delta \equiv (X' = \{D_1, D_2, \ldots, D_N\}, \mathscr{P}(X'), f^c_{(\cdot)}), S(P))$ such that $\int_\Omega |\bar{f}'_{\{D_k\}}(\omega) - f^c_{\{D_k\}}(\omega)| P(d\omega) \leqslant \Delta/N$ $(1 \leqslant \forall k \leqslant N)$. Here it is clear that the imaginary C^*-measurement $\mathbf{M}_{C_0(\Omega)}(\mathbf{O}^c_\Delta, S(P))$ approximates to $\mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$ if Δ is sufficiently small.

Remark 4.8. The above arguments bridge the gap between Methods 1 and 2. Also, it should be noted that Theorem 3.6 connects Axiom 1 with Method 1. This is quite important. That is because we consider that all results in this paper should be consequences of "objective fuzzy measurement theory".

Example 4.9 (*The subjective probability in the measurement of pencil's length*). Consider the (objective and subjective) C^*-measurement $\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}}(\rho_0))$ as in Example 3.11. Note that $\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}}(\rho_0)) = \mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S(\rho_0))$ from the subjective point of view. Let $\mathbf{M}_{L^\infty(\mathbf{R},\rho_0)}(\bar{\mathbf{O}}_{\mathscr{Z}} \equiv (\mathbb{Z}, \mathscr{P}(\mathbb{Z}), \bar{\zeta}_{(\cdot)}), S(1))$ be the normalized W^*-algebraic representation of $\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S(\rho_0))$. Here we see, by Method 2, that

(∗) the subjective probability that the measured value z $(\in \mathbb{Z})$ by $\mathbf{M}_{L^\infty(\mathbf{R};\rho_0)}(\bar{\mathbf{O}}_{\mathscr{Z}}, S(1))$ belongs to Ξ $(\in \mathscr{P}(\mathbb{Z}))$ is given by ${}_{L^1(\mathbf{R};\rho_0)}\langle 1, \bar{\zeta}_\Xi \rangle_{L^\infty(\mathbf{R};\rho_0)} = \int_{\mathbf{R}} \bar{\zeta}_\Xi(\omega)\ \rho_0(d\omega) = \frac{1}{10}\int_{10}^{20} \bar{\zeta}_\Xi(\omega) m(d\omega)$.

Therefore, if $\Xi = \{x\}$,

$$ {}_{L^1(\mathbf{R};\rho_0)}\langle 1, \bar{\zeta}_\Xi \rangle_{L^\infty(\mathbf{R};\rho_0)} = \begin{cases} \frac{1}{20} & \text{if } x = 10, 20, \\ \frac{1}{10} & \text{if } x = 11, 12, \ldots, 18, 19, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4} $$

which is of course the same as the result (3.7) of Example 3.11.

Now we introduce "quasi-product observables" in the W^*-algebraic formulation.

Definition 4.10 (*Quasi-product W^*-observable*). Let $\mathscr{N}$ be a W^*-algebra. Let K be a set. Let $\{\bar{\mathbf{O}}_k \equiv (X_k, \mathscr{F}_k, F_k)\colon k \in K\}$ is a family of W^*-observables in $\mathscr{N}$. Let $(\times_{k\in K} X_k, \times_{k\in K} \mathscr{F}_k)$ be the product measurable space of $\{(X_k, \mathscr{F}_k)\colon k \in K\}$. A W^*-observable $\bar{\mathbf{O}} = (\times_{k\in K} X_k, \times_{k\in K} \mathscr{F}_k, \hat{F})$ in $\mathscr{N}$ is called a quasi-product W^*-observable with marginal W^*-observables $\{\bar{\mathbf{O}}_k\colon k \in K\}$, if $\bar{\mathbf{O}}$ satisfies the following condition (∗):

(∗) for each $k_0 \in K$, the k_0th marginal observable of $\bar{\mathbf{O}}$ is equal to the observable $\bar{\mathbf{O}}_{k_0}$, that is, it satisfies that $\hat{F}(\times_{k\in K} \Xi_k) = F_{k_0}(\Xi_{k_0})$ $(\forall \Xi_{k_0} \in \mathscr{F}_{k_0})$, where $\Xi_k = \Xi_{k_0} (k = k_0) = X_k$ $(k \neq k_0)$ in $\times_{k\in K} \Xi_k$.

The quasi-product W^*-observable $\bar{\mathbf{O}} \equiv (\times_{k\in K} X_k, \times_{k\in K} \mathscr{F}_k, \hat{F})$ (with marginal W^*-observables $\{\bar{\mathbf{O}}_k\colon k \in K\}$) is sometimes denoted by $(\times_{k\in K} X_k, \times_{k\in K} \mathscr{F}_k, \overset{\bar{\mathbf{O}}}{\times}_{k\in K} F_k)$, or $\overset{\bar{\mathbf{O}}}{\times}_{k\in K} \bar{\mathbf{O}}_k$.

Here we can state the following methods (Methods 2′ and 2″) as a consequence of Method 2 (cf. Remark 1.14).

Method 2′ (Subjective simultaneous measurement). *Consider a simultaneous W^*-measurement $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$ where $\bar{\mathbf{O}} \equiv (\times_{k\in K} X_k, \times_{k\in K} \mathscr{F}_k, \overset{\bar{\mathbf{O}}}{\times}_{k\in K} F_k)$ is the quasi-product W^*-observable in $\mathscr{N}$. Assume that $x(= (x_k)_{k\in K} \in \times_{k\in K} X_k)$ is a measured value obtained by the W^*-measurement $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$. Then we consider*

(i) *the imaginary probability that the $x(= (x_k)_{k\in K})$ belongs to a set $\hat{\Xi}(\in \times_{k\in K} \mathscr{F}_k)$ is given by $\bar{\rho}((\overset{\bar{\mathbf{O}}}{\times}_{k\in K} F_k)(\hat{\Xi}))$,*

(ii) *the x_k, $k \in K$, is regarded as the value of the observable $\bar{\mathbf{O}}_k$ for the system $S(\bar{\rho})$.*

The following is a consequence of the identification: "measurement" = "inference" (cf. Remark 1.14 or [9]). Also, by a similar reason mentioned in Remark 1.13, we sometimes identify $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(*))$ with $\bar{\mathbf{O}}$.

Method 2″ (Subjective inference). *Let S be a fuzzy system with the imaginary state $\bar{\rho}(\in \mathfrak{S}^n(\mathscr{N}_*))$, which is formulated in a W^*-algebra $\mathscr{N}$. Let $\bar{\mathbf{O}} \equiv (X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2, F_1 \overset{\bar{\mathbf{O}}}{\times} F_2)$ be a quasi-product W^*-observable with the marginals $\bar{\mathbf{O}}_1 \equiv (X_1, \mathscr{F}_1, F_1)$ and $\bar{\mathbf{O}}_2 \equiv (X_2, \mathscr{F}_2, F_2)$. Then,*

(∗) *when we know the value $x_1(\in X_1)$ of the W^*-observable $\bar{\mathbf{O}}_1$ for the system $S(\bar{\rho})$ [resp. S], we can infer, by the subjective inference $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S)(\approx \bar{\mathbf{O}})$ [resp. $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$], that the imaginary probability that*

$x_2 (\in X_2)$, the value of the observable $\bar{\mathbf{O}}_2$ for the system $S(\bar{\rho})$ [resp. S], belongs to a set $\Xi_2 (\in \mathscr{F}_2)$ is given by $P(x_1, \Xi_2)$, where

$$P(x_1, \Xi_2) = \lim_{\substack{\Xi_1 \to \{x_1\} \\ \mathscr{F}_1 \ni \Xi_1 \ni x_1}} \frac{\bar{\rho}((F_1 \overset{\bar{\mathbf{O}}}{\times} F_2)(\Xi_1 \times \Xi_2))}{\bar{\rho}((F_1 \overset{\bar{\mathbf{O}}}{\times} F_2)(\Xi_1 \times X_2))}. \tag{4.5}$$

This (4.5) is of course the symbolic representation. For the mathematical definition of conditional probability, see [7] or [8].

The following theorem is essential for the definition of "measurement error" (cf. Definition 4.12). Also, recall Remark 1.12.

Theorem 4.11. *Let $\mathscr{N}$ be a W^*-algebra. Let $\bar{\mathbf{O}}_1 \equiv (X_1, \mathscr{F}_1, F_1)$ and $\bar{\mathbf{O}}_2 \equiv (X_2, \mathscr{F}_2, F_2)$ be W^*-observables in $\mathscr{N}$ such that at least one of them is crisp. (So, without loss of generality, we assume that $\bar{\mathbf{O}}_2$ is crisp). Then, the following statements are equivalent*:

(i) *There exists a quasi-product observable $\bar{\mathbf{O}}_{12} \equiv (X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2, F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2)$ with marginals $\bar{\mathbf{O}}_1$ and $\bar{\mathbf{O}}_2$.*

(ii) $\bar{\mathbf{O}}_1$ *and* $\bar{\mathbf{O}}_2$ *commute, that is,* $F_1(\Xi_1)F_2(\Xi_2) = F_2(\Xi_2)F_1(\Xi_1)$ $(\forall \Xi_1 \in \mathscr{F}_1, \forall \Xi_2 \in \mathscr{F}_2)$.

Furthermore, if the above statements (i) *and* (ii) *hold, the uniqueness of $\bar{\mathbf{O}}_{12}$ is guaranteed. (So, we can write that $\bar{\mathbf{O}}_{12} = (X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2, F_1 \overset{\mathrm{u}}{\times} F_2) = \bar{\mathbf{O}}_1 \overset{\mathrm{u}}{\times} \bar{\mathbf{O}}_2$.)*

Proof. When $\bar{\mathbf{O}}_1 \equiv (X_1, \mathscr{F}_1, F_1)$ and $\bar{\mathbf{O}}_2 \equiv (X_2, \mathscr{F}_2, F_2)$ are both crisp observables, it is proved in [2]. By the same way, we can prove this theorem. It is clear that (ii) $\Rightarrow$ (i) since we can construct a W^*-observable $(X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2, H)$ such that $H(\Xi_1 \times \Xi_2) = F_1(\Xi_1)F_2(\Xi_2)$ $(\forall \Xi_1 \in \mathscr{F}_1, \forall \Xi_2 \in \mathscr{F}_2)$. Thus, it suffices to prove that (i) $\Rightarrow$ (ii). Assume that (i) holds. Let Ξ_1 and Ξ_2 be any element in $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. Put $\Xi_1^1 = \Xi_1$, $\Xi_1^2 = X_1 \backslash \Xi_1$, $\Xi_2^1 = \Xi_2$ and $\Xi_2^2 = X_2 \backslash \Xi_2$. Put $H = F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2$. Note that

$$0 \leqslant H(\Xi_1^i \times \Xi_2^j) \leqslant H(X_1 \times \Xi_2^j) \equiv F_2(\Xi_2^j) \ (= \text{“projection”}). \tag{4.6}$$

This implies that $H(\Xi_1^i \times \Xi_2^j)$ and $F_2(\Xi_2^j)$ commute, and so, $H(\Xi_1^i \times \Xi_2^j)$ and $I - F_2(\Xi_2^j)$ commute. Hence, $F_1(\Xi_1)$ $(= H(\Xi_1^1 \times \Xi_2^1) + H(\Xi_1^1 \times \Xi_2^2))$ and $F_2(\Xi_2)$ $(= F_2(\Xi_2^1))$ commute. Therefore, we get that (i) $\Rightarrow$ (ii).

Next we prove the uniqueness of H under the assumption (i) (and so (ii)). Note that $0 \leqslant H(\Xi_1^i \times \Xi_2^j) \leqslant H(\Xi_1^i \times X_2) \equiv F_1(\Xi_1^i)$. This implies, by the commutativity condition (ii) and (4.6), that

$$0 \leqslant H(\Xi_1^i \times \Xi_2^j) \leqslant F_2(\Xi_2^j) F_1(\Xi_1^i) F_2(\Xi_2^j) = F_1(\Xi_1^i) F_2(\Xi_2^j).$$

Therefore, we see that $I = \sum_{i,j=1,2} H(\Xi_1^i \times \Xi_2^j) \leqslant \sum_{i,j=1,2} F_1(\Xi_1^i) F_2(\Xi_2^j) = I$. Then, we obtain that $H(\Xi_1 \times \Xi_2) = F_1(\Xi_1) F_2(\Xi_2)$, that is, H is unique. Therefore, we finish the proof.

Now we shall introduce several statistical quantities in fuzzy theory. Consider a W^*-measurement $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}} \equiv (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, F), S(\bar{\rho}))$ in a W^*-algebra $\mathscr{N}$. Then, $\mathscr{E}(\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho})))$, the expectation of $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$, is of course defined by

$$\mathscr{E}(\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))) = \int_{\mathbf{R}} \lambda \bar{\rho}(F(\mathrm{d}\lambda)). \tag{4.7}$$

Also, $\mathrm{var}(\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho})))$, the variance of $\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))$, is defined by

$$\mathrm{var}(\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))) = \int_{\mathbf{R}} \left[\lambda - \mathscr{E}(\mathbf{M}_{\mathscr{N}}(\bar{\mathbf{O}}, S(\bar{\rho}))) \right]^2 \bar{\rho}(F(\mathrm{d}\lambda)). \tag{4.8}$$

Here, we can define "measurement error", which is one of our purposes.

Definition 4.12 (*Measurement error*). Consider two W^*-measurements $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}_1 \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F_1), S(\bar{\rho}))$ and $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}_2 \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F_2), S(\bar{\rho}))$ in a W^*-algebra $\mathcal{N}$. And consider the simultaneous W^*-measurement $\mathbf{M}_{\mathcal{N}}\big(\bar{\mathbf{O}}_{12} \equiv (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2), S(\bar{\rho})\big)$. (The existence is not guaranteed in general). Then, $\Delta(\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}_{12}, S(\bar{\rho})))$, the measurement error of $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}_1, S(\bar{\rho}))$ for $\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}_2, S(\bar{\rho}))$, is defined by

$$\Delta(\mathbf{M}_{\mathcal{N}}(\bar{\mathbf{O}}_{12}, S(\bar{\rho}))) = \left[\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \bar{\rho}\big((F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2)(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\big)\right]^{1/2}.$$

Of course this depends on the choice of the quasi-product observable $\bar{\mathbf{O}}_{12}$ in general. However, if at least one of them (i.e., $\bar{\mathbf{O}}_1$ and $\bar{\mathbf{O}}_2$) is crisp, the uniqueness is guaranteed by Theorem 4.11.

Now we can state the main example (i.e., the measurement error of pencil's length), which should be compared with the error in Heisenberg's uncertainty relation in Section 7. In this example we assume Bayes's postulate, which will be prepared in the next section (cf. Remark 5.3).

Example 4.13 (*The measurement error of pencil's length*). Let $\Omega \equiv \{\omega_1, \omega_2, \dots, \omega_N\}$ be a set of pencils with various lengths. (If the physical foundation is required, it suffices to consider that $\Omega \subset \hat{\Omega}$ where $\hat{\Omega}$ is a quite large phase space in which these pencils are formulated. Cf. Remark 3.9.) Define the map $\phi : \Omega \to \mathbf{R}$ such that $\phi(\omega_n) =$ "the length of a pencil ω_n", which induces the C^*-homomorphism Φ: $\overline{C_0(\mathbf{R})} \to C(\Omega)$ such that $\overline{C_0(\mathbf{R})} \ni f(\cdot) \overset{\Phi}{\mapsto} f(\phi(\cdot)) \in C(\Omega)$. Assume the equal weight v_u on Ω (cf. Remark 5.3). Then, we can define the image measure P on $\mathbf{R}$ of v_u (i.e., $P(D) = v_u(\phi^{-1}(D))$ ($\forall D \in \mathcal{B}_{\mathbf{R}}$), that is, $P = \Phi^* v_u$). Here, it should be noted that the P ($\in \mathcal{M}_{+1}(\mathbf{R})$) has the subjective aspect since v_u is so. Under the hypothesis that N is sufficiently large, we assume that $P(D) = \int_D \bar{\rho}(x) m(\mathrm{d}x)$ where m is the Lebesgue measure on $\mathbf{R}$. Recall fuzzy numbers observables $\mathbf{O}_{\mathscr{Z}} = (\mathbb{Z}, \mathcal{P}_0(\mathbb{Z}), \zeta_{(\cdot)})$ in $C_0(\mathbf{R})$ and $\bar{\mathbf{O}}_{\mathscr{Z}} = (\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \bar{\zeta}_{(\cdot)})$ in $L^\infty(\mathbf{R}; m)$ (cf. Example 4.9). Then, we see, by (4.3), that $\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S(P)) \approx_{W^*} \mathbf{M}_{L^\infty(\mathbf{R};m)}(\bar{\mathbf{O}}_{\mathscr{Z}}, S(\bar{\rho}))$. Define the W^*-observable $\bar{\mathbf{O}}_{\mathscr{Z}}^{\mathbf{R}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \bar{\zeta}^{\mathbf{R}}_{(\cdot)})$ in $L^\infty(\mathbf{R}; m)$ such that $\bar{\zeta}^{\mathbf{R}}_{\Xi}(x) = \bar{\zeta}_{\Xi \cap \mathbb{Z}}(x)$ ($\forall x \in \mathbf{R}, \forall \Xi \in \mathcal{B}_{\mathbf{R}}$), which is clearly identified with $\bar{\mathbf{O}}_{\mathscr{Z}}$. Here consider the W^*-measurements $\mathbf{M}_{L^\infty(\mathbf{R};m)}(\bar{\mathbf{O}}_{\mathscr{Z}}^{\mathbf{R}}; S(\bar{\rho}))$ and $\mathbf{M}_{L^\infty(\mathbf{R};m)}(\bar{\mathbf{O}}_{\mathrm{FND}}, S(\bar{\rho}))$, where $\bar{\mathbf{O}}_{\mathrm{FND}}$ is the fundamental observable in $L^\infty(\mathbf{R}; m)$ (cf. Example 4.3). And consider the simultaneous W^*-measurement $\mathbf{M}_{L^\infty(\mathbf{R};m)}(\bar{\mathbf{O}}_{\mathscr{Z}}^{\mathbf{R}} \overset{\mathbf{u}}{\times} \bar{\mathbf{O}}_{\mathrm{FND}}, S(\bar{\rho}))$ (cf. Theorem 4.11). Here, assume, for convenience sake, that $\bar{\rho}(x) = \frac{1}{10}$ ($10 \leqslant \forall x \leqslant 20$), $= 0$ (otherwise). Then we see that

$$\begin{aligned}
&\Delta(\mathbf{M}_{L^\infty(\mathbf{R};m)}(\bar{\mathbf{O}}_{\mathscr{Z}}^{\mathbf{R}} \overset{\mathbf{u}}{\times} \bar{\mathbf{O}}_{\mathrm{FND}}, S(\bar{\rho})))\\
&= \left[\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \left(\int_{\mathbf{R}} \bar{\zeta}^{\mathbf{R}}_{\mathrm{d}\lambda_1}(x) \cdot \chi_{\mathrm{d}\lambda_2}(x) \bar{\rho}(x) m(\mathrm{d}x)\right)\right]^{1/2}\\
&= \left[\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |\lambda_1 - x|^2 \bar{\rho}(x) \bar{\zeta}^{\mathbf{R}}_{\mathrm{d}\lambda_1}(x)\right) m(\mathrm{d}x)\right]^{1/2}\\
&= \left[\frac{1}{10}\int_{10}^{20} \left(\sum_{n=10}^{20} |n - x|^2 \zeta_{\{n\}}(x)\right) m(\mathrm{d}x)\right]^{1/2} = \frac{1}{\sqrt{6}} = 0.4082\cdots.
\end{aligned}$$

Now we show the strong law of large numbers from the fuzzy theoretical point of view. This suggests that Kolmogorov's theory is one of the mathematical aspects of fuzzy measurement theory.

Theorem 4.14 (The "fuzzy theoretical" strong law of large numbers). *Let* $\mathbf{M}_{\mathcal{N}}$ $((\mathbf{R}^{\mathbb{N}}, \mathscr{B}_{\mathbf{R}}^{\mathbb{N}}, \overset{\mathbf{O}}{\times}_{n\in\mathbb{N}} F_n), S(\bar{\rho}))$ *be a* W^**-measurement. Put* $v(\hat{\Xi}) = \bar{\rho}((\overset{\mathbf{O}}{\times}_{n\in\mathbb{N}} F_n)(\hat{\Xi}))$ $(\forall \hat{\Xi} \in \mathscr{B}_{\mathbf{R}}^{\mathbb{N}})$ *and* $v_n(\Xi) = v((\times_{k=1}^{n-1} \mathbf{R}) \times \Xi \times (\times_{k=n+1}^{\infty} \mathbf{R}))$ $(\forall n \in \mathbb{N}, \forall \Xi \in \mathscr{B}_{\mathbf{R}})$. *Assume the following conditions* (i)–(iii):

(i) (*independency*), $v(\times_{n\in\mathbb{N}} \Xi_n) = \prod_{n\in\mathbb{N}} v_n(\Xi_n)$ *where* $\Xi_n = \mathbf{R}$ *except for a finite number of* n,

(ii) (*identical distribution*), $v_{n_1}(\Xi) = v_{n_2}(\Xi)$ $(\forall n_1, n_2 \in \mathbb{N}, \forall \Xi \in \mathscr{B}_{\mathbf{R}})$,

(iii) (L^1*-condition*), $\int_{\mathbf{R}^{\mathbb{N}}} x_1 v(d\hat{x}) = \gamma$ (*where* $\hat{x} = (x_1, x_2, \ldots) \in \mathbf{R}^{\mathbb{N}}$) *is finite.*

Then, we have the following statement:

($*$) *the imaginary probability that* $\hat{x} = (x_1, x_2, \ldots)$ $(\in \mathbf{R}^{\mathbb{N}})$, *the measured value obtained by the* W^**-measurement* $\mathbf{M}_{\mathcal{N}}((\mathbf{R}^{\mathbb{N}}, \mathscr{B}_{\mathbf{R}}^{\mathbb{N}}, \overset{\bar{\mathbf{O}}}{\times}_{n\in\mathbb{N}} F_n), S(\bar{\rho}))$, *belongs to* $\hat{\Xi}_\gamma \equiv \{\hat{x} \in \mathbf{R}^{\mathbb{N}} \mid \lim_{N\to\infty} \frac{\sum_{n=1}^{N} x_n}{N} = \gamma\}$ *is given by* 1.

Proof. Note that "the statement ($*$)" $\Leftrightarrow$ "$v(\hat{\Xi}_\gamma) = 1$". Therefore, this theorem is an immediate consequence of the usual strong law of large numbers for the probability space $(\mathbf{R}^{\mathbb{N}}, \mathscr{B}_{\mathbf{R}}^{\mathbb{N}}, v)$.

Example 4.15 (*The strong law of large numbers*). Let $\mathbf{M}_{L^\infty(\Omega;\mu)}(\bar{\mathbf{O}} \equiv (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, \bar{f}_{(\cdot)}), S(\bar{\rho}))$ be a W^*-measurement in a commutative W^*-algebra $L^\infty(\Omega;\mu)$ such that $\int_{\mathbf{R}} \lambda(\int_\Omega \bar{f}_{d\lambda}(\omega)\bar{\rho}(\omega)\mu(d\omega)) = \gamma$. And let $\mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$ be the normalized W^*- measurement (of $\mathbf{M}_{L^\infty(\Omega;\mu)}(\bar{\mathbf{O}}, S(\bar{\rho}))$) in the (normalized) W^*-algebra $L^\infty(\Omega;P)$ where $P(D) = \int_D \bar{\rho}(\omega)\mu(d\omega)$. Consider the infinite-dimensional product probability space $(\Omega^{\mathbb{N}}, \mathscr{B}^{\mathbb{N}}, \otimes_{n\in\mathbb{N}} P)$. Define the quasi-product W^*-observable $\bigotimes_{n\in\mathbb{N}} \bar{\mathbf{O}} = (\mathbf{R}^{\mathbb{N}}, \mathscr{B}_{\mathbf{R}}^{\mathbb{N}}, \bigotimes_{n\in\mathbb{N}} \bar{f})$ in $L^\infty(\Omega^{\mathbb{N}}; \otimes_{n\in\mathbb{N}} P)$ such that $(\bigotimes_{n\in\mathbb{N}} \bar{f})_{\times_{n\in\mathbb{N}}\Xi_n}(\omega_1, \omega_2, \ldots) = \prod_{n\in\mathbb{N}} \bar{f}_{\Xi_n}(\omega_n)$ (a.e. $\otimes_{n\in\mathbb{N}} P$) where $\Xi_n = \mathbf{R}$ except for a finite numbers of n. Then, we get the simultaneous W^*-measurement $\mathbf{M}_{L^\infty(\Omega^{\mathbb{N}}, \otimes_{n\in\mathbb{N}} P)}(\bigotimes_{n\in\mathbb{N}} \bar{\mathbf{O}}, S(\bigotimes_{n\in\mathbb{N}} 1))$, which is also equal to the repeated W^*-measurement of $\bigotimes_{n\in\mathbb{N}} \mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$'s. It is clear that this satisfies the conditions (i)–(iii) in Theorem 4.14. Therefore, we get the statement ($*$) in Theorem 4.14 for the W^*-measurement $\mathbf{M}_{L^\infty(\Omega^{\mathbb{N}}, \otimes_{n\in\mathbb{N}} P)}$ $(\bigotimes_{n\in\mathbb{N}} \bar{\mathbf{O}}, S(\bigotimes_{n\in\mathbb{N}} 1))$.

Example 4.16 (*Coin-tossing problem*). Consider the probability space $(\Omega, \mathscr{P}(\Omega), P)$ such that $\Omega = \{\text{face}, \text{tail}\}$ and $P(\{\text{face}\}) = \mathrm{P}(\{\text{tail}\}) = \frac{1}{2}$. Define the crisp W^*-observable $\bar{\mathbf{O}} = (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, \bar{f}_{(\cdot)})$ in $L^\infty(\Omega;P)$ such that $\bar{f}_\Xi(\text{face}) = 1$ (if $1 \in \Xi$), $= 0$ (if $1 \notin \Xi$) and $\bar{f}_\Xi(\text{tail}) = 1$ (if $0 \in \Xi$), $= 0$ (if $0 \notin \Xi$). Thus, we get the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$ and its repeated W^*-measurement $\bigotimes_{k=1}^{\infty} \mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$ as in the above example. Hence, we see that

($*$) the imaginary probability that $\hat{x}$ $(\in \mathbf{R}^{\mathbb{N}})$, the measured value obtained by the repeated W^*-measurement $\bigotimes_{k=1}^{\infty} \mathbf{M}_{L^\infty(\Omega;P)}(\bar{\mathbf{O}}, S(1))$, belongs to a set $\hat{\Xi}_{1/2} \equiv \{\hat{x} = (x_1, x_2, \ldots.) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = 1/2\}$ is given by 1.

5. Subjective method of membership functions

Since we can directly see "membership functions" (="fuzzy sets") in fuzzy measurement theory, it is easily expected that membership functions play an important role in this theory. In this section we propose the subjective method of membership functions (i.e., density function method). And we formulate "Shannon's entropy" and "Bayes's posulate" in this subjective method. That is because we believe that important concepts should be formulated in terms of "measurement".

Let $\mathbf{M}_{C_0(\Omega)}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), f_{(\cdot)}), S(\rho_0))$ be an imaginary C^*-measurement in a commutative C^*-algebra $C_0(\Omega)$. Here note that the subjective state ρ_0 $(\in \mathscr{M}_{+1}(\Omega))$ is determined by an observer. And therefore, another may consider a different (subjective) state. Let $\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}} \equiv (X, \mathscr{P}(X), \bar{f}_{(\cdot)}), S(1))$ be its normalized W^*-algebraic representation. Now we have the following question.

(∗) How can the observer guess the new (subjective) state $\bar{\rho}_x$ $(\in L^1_{+1}(\Omega;\rho_0))$ of the system S under the hypothesis that he gets the measured value x by the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}, S(1))$?

We solve this question in the following thought experiment. Let $\bar{\mathbf{O}}_{\mathrm{FND}} = (\Omega, \mathscr{B}_\Omega, \chi_{(\cdot)})$ be the fundamental observable in $L^\infty(\Omega;\rho_0)$ as in Example 4.3. Now consider the iterated measurement (i.e., "series measurement") of the observable $\bar{\mathbf{O}}$ and $\bar{\mathbf{O}}_{\mathrm{FND}}$ (that is, firstly the measurement of $\bar{\mathbf{O}}$ is taken, and next the measurement of $\bar{\mathbf{O}}_{\mathrm{FND}}$ is taken). The iterated measurement is formulated by the simultaneous measurement $\mathbf{M}_{L^\infty(\Omega;\rho_0)}$ $(\bar{\mathbf{O}} \overset{\mathrm{u}}{\times} \bar{\mathbf{O}}_{\mathrm{FND}}, S(1))$ where $\bar{\mathbf{O}} \overset{\mathrm{u}}{\times} \bar{\mathbf{O}}_{\mathrm{FND}}$ is defined by $(X \times \Omega, \mathscr{P}(X) \times \mathscr{B}_\Omega,\ \bar{h} \equiv \bar{f} \overset{\mathrm{u}}{\times} \chi)$ (in $L^\infty(\Omega;\rho_0)$) such that $\bar{h}_{\Xi_1 \times \Xi_2}(\omega) = \bar{f}_{\Xi_1}(\omega) \cdot \chi_{\Xi_2}(\omega)$ $(\forall \Xi_1 \in \mathscr{P}(X), \forall \Xi_2 \in \mathscr{B}_\Omega, \forall \omega \in \Omega)$. (The uniqueness is guaranteed by Theorem 4.11.) Then, we see, by Method 2″, that

(∗∗) when the observer knows the measured value x $(\in X)$ of the W^*-observable $\bar{\mathbf{O}}$ for the system S, he infers, by the (subjective) inference $\mathbf{M}_{L^\infty(\Omega;\rho_0)}$ $(\bar{\mathbf{O}} \overset{\mathrm{u}}{\times} \bar{\mathbf{O}}_{\mathrm{FND}}, S(1))$, that the imaginary probability that y $(\in \Omega)$, the value of the W^*-observable $\bar{\mathbf{O}}_{\mathrm{FND}}$ for the system S, belongs to a set Γ $(\in \mathscr{B}_\Omega)$ is given by $P(x,\Gamma)$, where

$$P(x,\Gamma) = \frac{\int_\Omega \bar{f}_{\{x\}}(\omega)\chi_\Gamma(\omega)\rho_0(d\omega)}{\int_\Omega \bar{f}_{\{x\}}(\omega)\rho_0(d\omega)} = \int_\Omega \left(\frac{\bar{f}_{\{x\}}(\omega)}{\int_\Omega \bar{f}_{\{x\}}(\omega)\rho_0(d\omega)} \right) \chi_\Gamma(\omega)\rho_0(d\omega). \tag{5.1}$$

Therefore, he will consider that the new (subjective) state $\bar{\rho}_x$ $(\in L^1_{+1}(\Omega;\rho_0))$ of the system S is equal to

$$\bar{\rho}_x(\omega) = \frac{\bar{f}_{\{x\}}(\omega)}{\int_\Omega \bar{f}_{\{x\}}(\omega)\rho_0(d\omega)} \qquad (\text{a.e. } \rho_0). \tag{5.2}$$

That is because the imaginary probability that the measured value by the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;\rho_0)}$ $(\bar{\mathbf{O}}_{\mathrm{FND}}, S(\bar{\rho}_x))$ belongs to Γ $(\in \mathscr{B}_\Omega)$ is also given by (5.1). Namely, the following reduction of density function occurs:

$$L^1_{+1}(\Omega;\rho_0) \ni 1 \mapsto \bar{\rho}_x(\omega) \in L^1_{+1}(\Omega;\rho_0) \tag{5.3}$$

by the fact (or information) that the observer gets the measured value x $(\in X)$ by $\mathbf{M}_{L^\infty(\Omega;\rho_0)}$ $(\bar{\mathbf{O}}, S(1))$. This of course occurs in his brain. Note that (5.2) has the form such as

$$\text{"density function"} = \frac{\text{"membership function"}}{\text{"normal factor (with respect to subjective state } \rho_0\text{)"}}. \tag{5.4}$$

Of course this is not all of density functions. For example, recall Example 4.13, in which the density function is created under the hypothesis that the Lebesgue measure on $\mathbf{R}$ is quite natural.

Example 5.1 (*The reduction of density function in the measurement of pencil's length*). Consider the W^*-measurement $\mathbf{M}_{L^\infty(\mathbf{R};\rho_0)}(\bar{\mathbf{O}}_{\mathscr{Z}} \equiv (\mathbb{Z}, \mathscr{P}(\mathbb{Z}), \bar{\zeta}), S(1))$ as in Example 4.9. Assume that we get the measured-value x $(\in \mathbb{Z})$. (Note that its imaginary probability is given by (4.4)). Then, by (5.3), we see the following reduction of the density function:

$$L^1_{+1}(\mathbf{R};\rho_0) \ni 1 \mapsto \bar{\rho}_x(\omega) = \frac{10\mathscr{Z}(\omega - x)}{\int_{10}^{20} \mathscr{Z}(\omega - x)m(d\omega)} \in L^1_{+1}(\mathbf{R};\rho_0). \tag{5.5}$$

Here, $\bar{\rho}_x(\omega) = 10\mathscr{Z}(\omega - x)$ if $x = 11, 12, \ldots, 19$, $= 20\mathscr{Z}(\omega - x)$ if $x = 10$ or 20. If we assume that the W^*-measurement $\mathbf{M}_{L^\infty(\mathbf{R};\rho_0)}(\bar{\mathbf{O}}_{\mathscr{Z}}, S(1))$ is the subjective aspect of the (objective and subjective) C^*-measurement

$\mathbf{M}_{C_0(\mathbf{R})}(\mathbf{O}_{\mathscr{Z}}, S_{\delta_{l_0}}(\rho_0))$, the actual measured value x is of course only 14 or 15 as in Example 3.11. However, it should be noted that we do not know the pure state δ_{l_0} in the above situation.

As one of the applications (of the reduction of a density function), we now study the "entropy" of the measurement. Here we have the following definition.

Definition 5.2 (*The entropy of measurement*). Without loss of generality (cf. Remark 4.7), consider a normalized W^*-measurement $\mathbf{M}_{L^\infty(\Omega;\rho_0)}$ $(\bar{\mathbf{O}} \equiv (X, \mathscr{P}(X), \bar{f}_{(\cdot)}),\ S(1))$ in a normalized commutative W^*-algebra $L^\infty(\Omega;\rho_0)$ (i.e., $\rho_0(\Omega)=1$), where the label set is assumed to be at most countable, i.e., $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. Then, the $H(\mathbf{M})$, the entropy of $\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}, S(1))$, is defined by

$$H(\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}, S(1)))$$
$$= \sum_{n=1}^{\infty} \int_\Omega \bar{f}_{\{x_n\}}(\omega)\rho_0(d\omega) \int_\Omega \frac{\bar{f}_{\{x_n\}}(\omega)}{\int_\Omega \bar{f}_{\{x_n\}}(\omega)\rho_0(d\omega)} \log \frac{\bar{f}_{\{x_n\}}(\omega)}{\int_\Omega \bar{f}_{\{x_n\}}(\omega)\rho_0(d\omega)} \rho_0(d\omega). \tag{5.6}$$

Particularly, when the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}, S(1))$ is the normalized W^*-algebraic representation of an imaginary C^*-measurement $\mathbf{M}_{C_0(\Omega)}(\mathbf{O} \equiv (X, \mathscr{P}_0(X), f_{(\cdot)}),\ S(\rho_0))$, the entropy $H(\mathbf{M}_{C_0(\Omega)}(\mathbf{O}, S(\rho_0)))$ is also defined by $H(\mathbf{M}_{L^\infty(\Omega;\rho_0)}\ (\bar{\mathbf{O}}, S(1)))$.

The definition is derived from the following consideration. Assume that we get the measured value x $(\in X)$ by the W^*-measurement $\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}, S(1))$. Note that its imaginary probability $P(x)$ is given by $P(x) = {}_{L^1(\Omega;\rho_0)}\langle 1, \bar{f}_{\{x\}}\rangle_{L^\infty(\Omega;\rho_0)} = \int_\Omega \bar{f}_{\{x\}}(\omega)\rho_0(d\omega)$. Also, we consider, by (5.2), that the new density function $\bar{\rho}_x$ is given by $\bar{\rho}_x(\omega) = \bar{f}_{\{x\}}(\omega)/\int_\Omega \bar{f}_{\{x\}}(\omega)\rho_0(d\omega)$, whose information quantity $I(x)$ is of course determined by $I(x) = \int_\Omega \bar{\rho}_x(\omega)\log\bar{\rho}_x(\omega)\rho_0(d\omega)$. Thus, the average information quantity, i.e., entropy, is given by $H(\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}, S(1))) = \sum_{n=1}^{\infty} P(x_n)\cdot I(x_n)$, which is clearly equal to (5.6). Also it should be noted that the formula (5.6) can easily be calculated as follows:

$$H(\mathbf{M}) = \sum_{n=1}^{\infty} \int_\Omega \bar{f}_{\{x_n\}}(\omega)\log \bar{f}_{\{x_n\}}(\omega)\rho_0(d\omega) - \sum_{n=1}^{\infty} P(x_n)\log P(x_n). \tag{5.7}$$

Therefore, if $\bar{\mathbf{O}}$ is crisp, we see that $H(\mathbf{M}) = -\sum_{n=1}^{\infty} P(x_n)\log P(x_n)$.

Remark 5.3 (*Bayes's postulate*). Let $\Omega \equiv \{\omega_1, \omega_2, \ldots, \omega_N\}$ be a finite set with the discrete topology. Consider the fundamental C^*-observable $\mathbf{O}_{\mathrm{FND}} \equiv (\Omega, \mathscr{P}(\Omega), \chi_{(\cdot)})$ in $C(\Omega)$, which is of course the finest observable in $C(\Omega)$. Let ρ_0^{m} be arbitrary subjective state (i.e., $\rho_0^{\mathrm{m}} \in \mathscr{M}_{+1}(\Omega)$). Then, by (5.7), we see that

$$H(\mathbf{M}_{C(\Omega)}(\mathbf{O}_{\mathrm{FND}}, S(\rho_0^{\mathrm{m}}))) = -\sum_{n=1}^{N} \rho_0^{\mathrm{m}}(\{\omega_n\})\log\rho_0^{\mathrm{m}}(\{\omega_n\}).$$

Also, it is well known that (i) $\sup\{H(\mathbf{M}_{C(\Omega)}(\mathbf{O}_{\mathrm{FND}}, S(\rho_0^{\mathrm{m}}))):\ \rho_0^{\mathrm{m}} \in \mathscr{M}_{+1}(\Omega)\} = \log N$, (ii) "$\rho_0^{\mathrm{m}}(\{\omega_n\}) = 1/N (\forall n)$" $\Leftrightarrow$ "$H(\mathbf{M}) = \log N$". Therefore, the equal weight ν_u (i.e., $\nu_u(D) = \sharp[D]/N$ $(\forall D \subseteq \Omega)$ is sometimes called *the state representing "knowing nothing"* (i.e., Bayes's postulate). That is because, if we know nothing for the system, we of course get maximal information by $\mathbf{M}_{C(\Omega)}(\mathbf{O}_{\mathrm{FND}}, S)$. This is the fuzzy theoretical explanation of "knowing nothing". All concepts should be defined in terms of "measurement". That is because we must always start from the principle (i.e., Axioms 0 and 1) and not the ambiguous word such as "knowing nothing".

Example 5.4 (*Crisp and fuzzy informations*). Let $\Omega \equiv \{\omega_1, \omega_2, \ldots, \omega_{100}\}$ be a set of pupils in some school. Let $\mathbf{O}_{\mathrm{b}} \equiv (X = \{y_{\mathrm{b}}, n_{\mathrm{b}}\}, \mathscr{P}(X), b_{(\cdot)})$ be the crisp C^*-observable in the commutative C^*-algebra $C(\Omega)$ such that $b_{\{y_b\}}(\omega_n) = 0$ (n is odd), $= 1$ (n is even), and $b_{\{n_b\}}(\omega_n) = 1 - b_{\{y_b\}}(\omega_n)$. Also, let $\mathbf{O}_f \equiv (Y = \{y_f, n_f\}, \mathscr{P}(Y), f_{(\cdot)})$ be the C^*-observable in C^*-algebra $C(\Omega)$ such that $f_{\{y_f\}}(\omega_n) = (n-1)/99$ ($\forall \omega_n \in \Omega$) and $f_{\{n_f\}}(\omega_n) = 1 - f_{\{y_f\}}(\omega_n)$. Let $\rho_0 \in \mathscr{M}_{+1}(\Omega)$, for example, assume that $\rho_0 = \nu_u$, i.e., the equal weight on Ω. Thus, $\nu_u(\{\omega_n\}) = 1/100$ ($\forall n$). Then, we see, by (5.7), that

$$H(\mathbf{M}_{C(\Omega)}(\mathbf{O}_b, S(\nu_u))) = -\|b_{\{y_b\}}\|_{L^1} \log \|b_{\{y_b\}}\|_{L^1} - \|b_{\{n_b\}}\|_{L^1} \log \|b_{\{n_b\}}\|_{L^1}$$

$$= -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = \log_2 2 = 1 \ \text{(bit)},$$

$$H(\mathbf{M}_{C(\Omega)}(\mathbf{O}_f, S(\nu_u))) = \int_\Omega f_{\{y_f\}}(\omega)\log f_{\{y_f\}}(\omega)\nu_u(d\omega) + \int_\Omega f_{\{n_f\}}(\omega)\log f_{\{n_f\}}(\omega)\nu_u(d\omega)$$

$$-\|f_{\{y_f\}}\|_{L^1}\log\|f_{\{y_f\}}\|_{L^1} - \|f_{\{n_f\}}\|_{L^1}\log\|f_{\{n_f\}}\|_{L^1}$$

$$\approx 2\int_0^1 \lambda \log_2 \lambda \, d\lambda + 1 = -\frac{1}{2\log_e 2} + 1 = 0.278\cdots \ \text{(bit)}.$$

For example, assume that the symbol "y_b" [resp. "n_b"] in X is interpreted by "boy" [resp. "girl"]. And "y_f" [resp. "n_f"] in Y is interpreted by "fast runner" [resp. "not fast runner"]. When we guess the pure state $(*)$ of the system S $(= S_{(*)}(\nu_u))$ in the above situation, it is proper that the crisp information "boy or girl" is more efficient than the fuzzy information "fast or not fast".

Remark 5.5 (*Fuzzy information theory*). "Shannon's entropy" is usually defined as follows (cf. [1]). Let $(\Omega, \mathscr{B}, P)$ be a probability space. Let $\mathbf{D} = \{D_1, D_2, \ldots\}$ be the countable decomposition of Ω. Then, the entropy $H(\mathbf{D})$ of $\mathbf{D}$ is defined by $H(\mathbf{D}) = -\sum_{n=1}^{\infty} P(D_n)\log P(D_n)$. Note that Definition 5.2 is the natural extension of Shannon's entropy if we regard the observable $\mathbf{O}$ as a "fuzzy decomposition" (cf. Remark 1.9). However, by the same reason mentioned in Remark 5.3 (or, as emphasized throughout this paper), we should never start from the mathematical concept "probability space".

6. Objective method of membership functions

In this section, we introduce "objective method of membership functions" (or in short, "membership function method"), which is characterized as the objective aspect of "subjective method of membership functions" (cf. Remark 6.4). And we formulate Zadeh's theory as the membership function method.

Let $\mathscr{A} \equiv C_0(\Omega)$ be a commutative C^*-algebra on some locally compact Hausdorff space Ω. Let $\mathbf{O}_1 \equiv (X, \mathscr{P}_0(X), f_{(\cdot)})$ and $\mathbf{O}_2 \equiv (Y, \mathscr{P}_0(Y), g_{(\cdot)})$ be C^*-observables in $C_0(\Omega)$. And let $\mathbf{O}_{12} \equiv (X \times Y, \mathscr{P}_0(X \times Y), h \equiv f \overset{\mathbf{O}_{12}}{\times} g)$ be the quasi-product observable with the marginals $\mathbf{O}_1$ and $\mathbf{O}_2$. Here note that the $\mathbf{O}_{12}$ is restricted by the condition (1.3) in Remark 1.12.

Let ρ_0 $(\in \mathscr{M}_{+1}(\Omega))$ be a (subjective) state of the system S. That is, an observer assumes that the system has the subjective state ρ_0. Consider the imaginary C^*- measurement $\mathbf{M}_{C_0(\Omega)}(\mathbf{O}_{12}, S(\rho_0))$ and its normalized W^*-algebraic representation $\mathbf{M}_{L^\infty(\Omega;\rho_0)}(\bar{\mathbf{O}}_{12}, S(1))$. And assume that the measured value is equal to (x, y) $(\in X \times Y)$. Thus, the observer considers, by (5.2), that the new state (i.e., new density function) $\bar{\rho}_{\{(x,y)\}}$ $(\equiv \bar{\rho}_{(x,y)} \in L^1_{+1}(\Omega, \rho_0))$ is equal to

$$\bar{\rho}_{\{(x,y)\}}(\omega) = \frac{h_{\{(x,y)\}}(\omega)}{\int_\Omega h_{\{(x,y)\}}(\omega)\rho_0(d\omega)}.$$

That is, the reduction "$L^1_{+1}(\mathbf{R};\rho_0) \ni 1 \mapsto \bar{\rho}_{\{(x,y)\}}(\omega) \in L^1_{+1}(\mathbf{R};\rho_0)$" occurs in observer's brain. More generally, it is reasonable to consider that

(1) [S] when the observer knows that the measured value belongs to $\Xi \times \Gamma$ ($\in \mathscr{P}_0(X \times Y)$), he considers that the new state $\bar{\rho}_{\Xi\times\Gamma}$ ($\in L^1_{+1}(\Omega;\rho_0)$) is equal to

$$\bar{\rho}_{\Xi\times\Gamma}(\omega) = \sum_{(x,y)\in\Xi\times\Gamma} \frac{\int_\Omega h_{\{(x,y)\}}(\omega)\rho_0(d\omega)}{\int_\Omega h_{\Xi\times\Gamma}(\omega)\rho_0(d\omega)} \cdot \bar{\rho}_{\{(x,y)\}}(\omega) = \frac{h_{\Xi\times\Gamma}(\omega)}{\int_\Omega h_{\Xi\times\Gamma}(\omega)\rho_0(d\omega)}.$$

Here recall (5.4), and note that "density function" is meaningless from the objective point of view (since ρ_0 is subjective, i.e., it depends on observer's knowledge). Therefore, from the objective point of view, the above subjective statement [S] implies (and is implied by) that

(1) [O] when the observer knows that the measured value belongs to $\Xi \times \Gamma$ ($\in \mathscr{P}_0(X \times Y)$), he considers that the new membership function is equal to $h_{\Xi\times\Gamma}$.

We believe that this statement [O] (i.e., the objective aspect of the subjective statement [S]) is the essence of Zadeh's theory. Of course the statement [O] is more fundamental than the above [S]. That is because there should always exist "objectivity" before "subjectivity".

In general, let $\mathbf{O} \equiv (\times_{k\in K} X_k, \mathscr{P}_0(\times_{k\in K} X_k), \overset{\mathbf{o}}{\times}_{k\in K} f^k)$ be any fixed quasi-product C^*-observable in a commutative C^*-algebra $C_0(\Omega)$. The C^*-observable $\mathbf{O}$ can be regarded as the following correspondence:

$$\mathscr{P}_0\left(\times_{k\in K} X_k\right) \ni \Xi_1 \times \Xi_2 \times \cdots \times \Xi_{|K|} \mapsto \left(\overset{\mathbf{o}}{\times}_{k\in K} f^k\right)_{\Xi_1\times\Xi_2\times\cdots\times\Xi_{|K|}} \in \overline{C_0(\Omega)}.$$

That is, the $\mathbf{O}$ determines the "fuzzy sets operation" such as

$$\text{"usual sets operation" on } \times_{k\in K} X_k \mapsto \text{"fuzzy sets operation" on } \Omega.$$

Example 6.1 (*Label, density function, membership function*). Consider the simultaneous measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_b \overset{\mathbf{u}}{\times} \mathbf{O}_f = (X \times Y, \mathscr{P}(X \times Y), (b \overset{\mathbf{u}}{\times} f)_{(\cdot)}), S(\rho_0))$, where $\mathbf{O}_b$ and $\mathbf{O}_f$ are as in Example 5.4 (crisp and fuzzy informations). Then, we get the following diagram:

(Label)		(Density function)		(Membership function)
BOY$(=\{y_b\}\times\{y_f,n_f\})$	$\cdots$	$\dfrac{b_{\{y_b\}}(\omega)}{\int_\Omega b_{\{y_b\}}(\omega)\rho_0(d\omega)}$	$\cdots$	$b_{\{y_b\}}(\omega)$
FAST$(=\{y_b,n_b\}\times\{y_f\})$	$\cdots$	$\dfrac{f_{\{y_f\}}(\omega)}{\int_\Omega f_{\{y_f\}}(\omega)\rho_0(d\omega)}$	$\cdots$	$f_{\{y_f\}}(\omega)$
BOY $\wedge$ FAST$(=\{y_b\}\times\{y_f\})$	$\cdots$	$\dfrac{b_{\{y_b\}}(\omega)f_{\{y_f\}}(\omega)}{\int_\Omega b_{\{y_b\}}(\omega)f_{\{y_f\}}(\omega)\rho_0(d\omega)}$	$\cdots$	$b_{\{y_b\}}(\omega)\cdot f_{\{y_f\}}(\omega)$
$\neg$BOY$(=\{n_b\}\times\{y_f,n_f\})$	$\cdots$	$\dfrac{b_{\{n_b\}}(\omega)}{\int_\Omega b_{\{n_b\}}(\omega)\rho_0(d\omega)}$	$\cdots$	$b_{\{n_b\}}(\omega)$

and so on. Here note again that "density function" is meaningless from the objective point of view since ρ_0 is subjective , i.e., it depends on observer's knowledge. From the objective point of view, "fuzzy sets operation" (i.e., the correspondence "label" $\mapsto$ "membership function") is essential.

Remark 6.2 ("*Extension principle*"). Without loss of generality (cf. Remark 1.3), assume that Ω_1 and Ω_2 are compact. Consider a continuous map $\phi : \Omega_1 \to \Omega_2$, which induces the C^*-homomorphism $\Phi : C(\Omega_2) \to C(\Omega_1)$

such that $C(\Omega_2) \ni f_2(\cdot) \overset{\Phi}{\mapsto} f_2(\phi(\cdot)) \in C(\Omega_1)$. Let f_1^0 be a membership function in $C(\Omega_1)$. We recommend the reader to calculate $\inf\{f_2(\omega_2): f_1^0 \leqslant \Phi f_2,\ 0 \leqslant f_2 \leqslant 1,\ f_2 \in C(\Omega_2)\}$. He will easily find "extension principle" (cf. [16]) in this exercise.

Remark 6.3. We must always consider a membership function as the element of an observable. This does not impose restrictions on our theory. That is because, for any membership function f $(\in \overline{C_0(\Omega)})$, we can always construct a C^*-observable $\mathbf{O} \equiv (\{y,n\}, \mathscr{P}(\{y,n\}), f_{(\cdot)})$ such that $f_{\{y\}}(\omega) = f(\omega)$ $(\forall \omega \in \Omega)$ and $f_{\{n\}}(\omega) = 1 - f(\omega)$.

Remark 6.4. In this section we introduce "objective method of membership functions" as a consequence of "subjective method of membership functions". However, it is clear that "objective method of membership functions" is independent of "subjective method of membership functions". That is, we can jump from Section 2 to this section. The reason that we did not do so is that "objectivity" and "subjectivity" should be appreciated in comparison with each other.

Remark 6.5. In quantum mechanics, "objectivity" and "subjectivity" are sometimes mixed (cf. Remark 4.6). Therefore, if we restrict ourselves to commutative case, we can sum up the actual relation between "objective fuzzy measurement theory" and "subjective fuzzy measurement theory" in the following diagram:

["objectivity"]		["subjectivity"]
principle, "true or not true"	$\leftrightarrow$	method, "useful or not useful"
individualistic, real	$\leftrightarrow$	statistical, imaginary
C^*-algebra, $C_0(\Omega)$, topology	$\leftrightarrow$	W^*-algebra, $L^\infty(\Omega;\mu)$, measure theory
membership function	$\leftrightarrow$	density function
Newtonian mechanics	$\leftrightarrow$	statistical mechanics.

(Also, see Remark 4.8.) Kolmogorov's theory is the mathematical theory concerning the probability space $(X, \mathscr{F}, P)$ induced by $\mathbf{M}_{\mathcal{N}}$ $((X, \mathscr{F}, F), S(\bar{\rho}))$ (i.e., $P(\cdot) = \bar{\rho}(F(\cdot))$). Therefore, "fuzzy sets" cannot be found in his theory. Also, Shannon's entropy (or more generally, "fuzzy information theory" in Remark 5.5) and Bayes's postulate (cf. Remark 5.3) belong to the category of "subjectivity". And furthermore, "fuzzy measure theory" (cf. Remark 3.12) is purely subjective, i.e., beyond "subjectivity". On the other hand, Zadeh's theory completely belongs to the category of "objectivity". Now we can achieve the second purpose of this paper, that is, these fuzzy theories are characterized as some aspects of fuzzy measurement theory. Namely, behind these theories, Axioms 0 and 1 exist. This is the reason that these theories are quite applicable in spite that they have no principle but only method (cf. Remark 3.3).

In this section we emphasized the objective aspect of Zadeh's theory. This may not be a usual opinion. In fact, his theory was first invented as the mathematical tool to analyze the ambiguity of the words. Therefore, some may consider that the essence of Zadeh's theory is "subjective". However, we believe that Zadeh's theory is objective ($\approx$ individualistic) as the method to analyze the ambiguity of the words.

As stated in [9], the "words" are not essential in our theory. That is because "observables" exist before "words". However, we think that we must mention something about the "words problem" in our theory. Thus, let us mention it. It is quite reasonable to consider that, in order to represent what he want to assert, the researcher can choose a suitable mathematical theory. The researchers (of fuzzy sets theory) do not choose "measure theory". In other words, they consider that "measure theory" is not proper for analyzing the ambiguity of the words. The reason is clear, that is, they are not satisfied with "statistical" statements such as "She is beautiful *in the average sense*". They want to analyze the ambiguity of the words from the "individualistic" point of view.

Now consider a tensor product commutative C^*-algebra $C(\Omega_1) \otimes C(\Omega_2) \equiv C(\Omega_1 \times \Omega_2)$. And consider a C^*-observable $\mathbf{O} \equiv (X = \{x_1, x_2, \ldots, x_n\}, \mathscr{P}(X), h_{(\cdot)})$ in $C(\Omega_1 \times \Omega_2)$. Assume that the $\mathbf{O}$ is objective, that is, it is the mathematical representation of a certain actual observable $\mathcal{O}_X$. For any fixed x_k $(\in X)$, the $h_{\{x_k\}}$ is of course regarded as a "fuzzy set" on $\Omega_1 \times \Omega_2$. Here note that $L \equiv \{g \in C(\Omega_2)\colon 0 \leqslant g(\omega_2) \leqslant 1 (\forall \omega_2 \in \Omega_2)$, i.e., g is a "fuzzy set" on $\Omega_2\}$ is a lattice. Therefore, for any fixed x_k $(\in X)$ and any fixed ω_1^0 $(\in \Omega_1)$, the $h_{\{x_k\}}(\omega_1^0, \cdot)$ can be regarded as a "fuzzy set" on Ω_2, i.e., an element of L. That is, the $h_{\{x_k\}}$ is also regarded as "L-fuzzy set" on Ω_1 (cf. [3]). Therefore, the C^*-observable $\mathbf{O}$ in the tensor algebra $C(\Omega_1 \times \Omega_2)$ may be called a *L-fuzzy observable.*

Also, consider a certain mixed state ρ_2^{m} $(\in \mathfrak{S}^{\mathrm{m}}(C(\Omega_2)^*) \equiv \mathscr{M}_{+1}(\Omega_2))$. And consider the "$C^*$-measurement" $\mathbf{M}_{C(\Omega_1 \times \Omega_2)}(\mathbf{O}, S(* \otimes \rho_2^{\mathrm{m}}))$. This may be called a "*probablistic C^*-observable in $C(\Omega_1)$*" since the pair $[h_{\{x_k\}}(*, *), \rho_2^{\mathrm{m}}]$ is a "probabilistic set on Ω_1" in the sense of [4]. Also, define the C^*-observable $\mathbf{O}_1 \equiv (X, \mathscr{P}(X), f_{(\cdot)})$ in $C(\Omega_1)$ such that $f_{\{x_k\}}(\omega_1) = {}_{C(\Omega_2)^*}\langle \rho_2^{\mathrm{m}}, h_{\{x_k\}}(\omega_1, \cdot)\rangle_{C(\Omega_2)}$ $(\forall x_k \in X, \forall \omega_1 \in \Omega_1)$, which should be called the *average C^*-observable* of the probabilistic C^*-observable $\mathbf{M}_{C(\Omega_1 \times \Omega_2)}(\mathbf{O}, S(* \otimes \rho_2^{\mathrm{m}}))$. The average C^*-observable $\mathbf{O}_1$ clearly has the subjective ($\approx$ statistical) aspect since the mixed state ρ_2^{m} is hidden in this C^*-observable.

Here consider the reverse of the above arguments. That is, let us start from the average C^*-observable $\mathbf{O}_1$ in $C(\Omega_1)$. The researchers (of fuzzy sets theory) may not be concerned with this observable $\mathbf{O}_1$ since they are not satisfied with "statistical" statements. They want to analyze the $\mathbf{O}_1$ more precisely. And, after some observations, they will find the "probabilistic C^*-observable" $\mathbf{M}_{C(\Omega_1 \times \Omega_2)}(\mathbf{O}, S(* \otimes \rho_2^{\mathrm{m}}))$ in $C(\Omega_1)$. However, the subjective state ρ_2^{m} may not concern them. And finally they reach the conclusion that the most important is the (objective) observable $\mathbf{O}$ ($\approx \mathcal{O}_X$) in a tensor algebra $C(\Omega_1 \times \Omega_2)$, which is also regarded as a L-fuzzy observable. This is the standard way in the research of fuzzy sets theory. Therefore, we assert that the researchers (of fuzzy sets theory) investigate the ambiguity of the words from the objective ($\approx$ individualistic) point of view. This is no wonder if we consider that many words were created in order to represent "observables".

7. Heisenberg's uncertainty relation

Most physicists confuse two "uncertainty relations" in quantum mechanics. Now let us explain this.

One is statistical uncertainty relation. Let Q and P be a position quantity and a momentum quantity, respectively (i.e. Q and P are self-adjoint operators on a Hilbert space V satisfying that $QP - PQ = i\hbar$, $\hbar$ is the Plank constant), and let $\bar{\rho}$ be a state (i.e. $\bar{\rho} \in \mathfrak{S}^n(B(V)_*) \equiv \mathrm{Tr}_{+1}(V)$) of a quantum system. Consider two W^*-measurements $\mathbf{M}_{B(V)}([Q] \equiv (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, E_Q), S(\bar{\rho}))$ and $\mathbf{M}_{B(V)}([P] \equiv (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, E_P), S(\bar{\rho}))$. The expectations are defined by $\mathscr{E}(\mathbf{M}_{B(V)}([Q], S(\bar{\rho}))) = \int_{\mathbf{R}} \lambda\, \bar{\rho}(E_Q(\mathrm{d}\lambda)) = \mathrm{tr}[\bar{\rho} \cdot Q]_V$ and $\mathscr{E}(\mathbf{M}_{B(V)}([P], S(\bar{\rho}))) = \mathrm{tr}[\bar{\rho} \cdot P]_V$ (cf. the formula (4.7)). Also, the variances are as follows (cf. the formula (4.8)):

$$\mathrm{var}(\mathbf{M}_{B(V)}([Q], S(\bar{\rho}))) = \int_{\mathbf{R}} |\lambda - \mathscr{E}(\mathbf{M}_{B(V)}([Q], S(\bar{\rho})))|^2 \bar{\rho}(E_Q(\mathrm{d}\lambda))$$

$$= \mathrm{tr}[\bar{\rho} \cdot (Q - \mathrm{tr}[\bar{\rho}Q]_V)^2]_V$$

and $\mathrm{var}(\mathbf{M}_{B(V)}([P], S(\bar{\rho}))) = \mathrm{tr}[\bar{\rho} \cdot (P - \mathrm{tr}[\bar{\rho}P]_V)^2]_V$. From this and a simple calculation, we can easily obtain the following uncertainty relation:

$$[\mathrm{var}(\mathbf{M}_{B(V)}([Q], S(\bar{\rho}))]^{\frac{1}{2}} \cdot [\mathrm{var}(\mathbf{M}_{B(V)}([P], S(\bar{\rho}))]^{\frac{1}{2}} \geqslant \frac{\hbar}{2}.$$

This is called the statistical uncertainty relation (discovered by Robertson [14] in 1929).

Another is the individualistic uncertainty relation, which was discovered by Heisenberg in 1927 using the famous thought experiment of γ-rays microscope. He asserted as follows:

(A) *The particle position q and momentum p can be measured "simultaneously", if the errors $\Delta(q)$ and $\Delta(p)$ in determining the particle position and momentum are permitted to be non-zero. Moreover, for any* $\varepsilon > 0$, *we can take the "simultaneous" measurement of the position q and momentum p such that* $\Delta(q) < \varepsilon$ (*or* $\Delta(p) < \varepsilon$).

(B) *However, the following Heisenberg's uncertainty relation holds*:

$$\Delta(q) \cdot \Delta(p) \approx \hbar$$

for all "simultaneous" measurements of the particle position and momentum.

We of course call it Heisenberg's uncertainty relation.

Most physicists confuse Heisenberg's uncertainty relation with statistical uncertainty relation. That is, statistical uncertainty relation is usually regarded as the mathematical representation of Heisenberg's uncertainty relation (though von Neumann commented on this gap in his famous book [13]). The "error" is clearly different from "$[\text{variance}]^{1/2}$". In [6], we pointed out this misunderstanding among physicists and proposed Proposition 1.1 (in Section 1) as the mathematical foundation of Heisenberg's uncertainty relation. However, we do not think that Proposition 1.1 is the final version of Heisenberg's uncertainty relation. That is because the meaning of the quantities $||(\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)||_{V\otimes U}$, $k = 1,2$, is not yet clear in Proposition 1.1.

Here note that Heisenberg's uncertainty relation includes "paradox" in itself. That is, it is "reasonable" to consider that "error" = |"true value" − "measured value"| and "true value" = "measured value by exact measurement". If it is true, the "error" in Heisenberg's uncertainty relation cannot be defined. That is because the statement (B) asserts that exact measurement does not exit. Therefore, the question "What is error ?" is significant. This is the original motivation that we propose "(objective and subjective) fuzzy measurement theory".

We begin with the following definition.

Definition 7.1 (*Approximate simultaneous measurement and its error*). Let V be a Hilbert space. Let A_1 and A_2 be self-adjoint operators (i.e., physical quantities) in a Hilbert space V. Put $[A_k] = (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, E_{A_k})$, i.e., the spectral measure of A_k. Then, a W^*-measurement $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_{12} \equiv (\mathbf{R}^2, \mathscr{B}_{\mathbf{R}^2}, F \equiv F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2), S(*))$ is called an approximate simultaneous measurement of A_1 and A_2, if it satisfies the following conditions (i) and (ii):

(i) for each $k = 1,2$, the kth marginal W^*-observable $\bar{\mathbf{O}}_k \equiv (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, F_k)$ (of $\bar{\mathbf{O}}_{12}$) and the crisp W^*-observable $[A_k] \equiv (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, E_{A_k})$ commute,

(ii) for each $k = 1,2$, $\mathscr{E}(\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_k, S(|\psi\rangle\langle\psi|))) = \mathscr{E}(\mathbf{M}_{B(V)}([A_k], S(|\psi\rangle\langle\psi|)))$ holds, that is,

$$\int_{\mathbf{R}} x_k \langle \psi, F_k(\mathrm{d}x_k)\psi\rangle_V = \int_{\mathbf{R}} \lambda \langle \psi, E_{A_k}(\mathrm{d}\lambda)\psi\rangle_V \quad (= \langle \psi, A_k\psi\rangle_V) \quad (\forall \psi \in \mathrm{Dom}(A_k)).$$

Also, the kth measurement error of the approximate simultaneous measurement $\mathbf{M}_{B(V)}$ $(\bar{\mathbf{O}}_{12}, S(\bar{\rho}))$ of A_1 and A_2 is defined by $\Delta(\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_k \overset{\mathrm{u}}{\times} [A_k], S(\bar{\rho})))$ (cf. Definition 4.12).

Here we prepare two lemmas.

Lemma 7.2. *Let A_1 and A_2 be self-adjoint operators in a Hilbert space V. Let $[\hat{A}_1, \hat{A}_2, U, \phi_0]$ be an approximate simultaneous measurement of A_1 and A_2 in the sense of Proposition* 1.1. *Put* $[\hat{A}_k] = (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, E_{\hat{A}_k})$. *Define the W^*-observable* $\bar{\mathbf{O}}_{12} \equiv (\mathbf{R}^2, \mathscr{B}_{\mathbf{R}^2}, F \equiv F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2)$ *such that*

$$\langle \psi_1, F(\Xi_1 \times \Xi_2)\psi_2\rangle_V = \left\langle \psi_1 \otimes \phi_0, \left(\left(E_{\hat{A}_1} \overset{\mathrm{u}}{\times} E_{\hat{A}_2}\right)(\Xi_1 \times \Xi_2)\right)(\psi_2 \otimes \phi_0)\right\rangle_{V\otimes U} \tag{7.1}$$

$$(\forall \psi_1, \forall \psi_2 \in V, \forall \Xi_1 \in \mathscr{B}_{\mathbf{R}}, \forall \Xi_2 \in \mathscr{B}_{\mathbf{R}}).$$

(Here note that the uniqueness of $E_{\hat{A}_1} \overset{\mathbf{u}}{\times} E_{\hat{A}_2}$ *is guaranteed by Theorem* 4.11). *Then, we see*

(i) *the* W^**-measurement* $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_{12}, S(*))$ *is an approximate simultaneous measurement of* A_1 *and* A_2 *in the sense of Definition* 7.1.

(ii) $\Delta(\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k], S(|\psi\rangle\langle\psi|))) = \|\, (\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)\, \|_{V\otimes U}$ *for all* $\psi \in \mathrm{Dom}(\mathrm{A_k})$.

Proof. Note that (7.1) implies that $\langle \psi_1, F_k(\Xi)\psi_2\rangle_V = \langle \psi_1 \otimes \phi_0, E_{\hat{A}_k}(\Xi)(\psi_2 \otimes \phi_0)\rangle_{V\otimes U}$ $(\forall \psi_1, \psi_2 \in V, \forall \Xi \in \mathscr{B}_{\mathbf{R}})$. Therefore we see, from the commutativity of $E_{\hat{A}_k}$ and $E_{A_k \otimes I}$, that

$$\langle \psi_1, F_k(\Xi)E_{A_k}(\Xi')\psi_2\rangle_V = \langle \psi_1 \otimes \phi_0, E_{\hat{A}_k}(\Xi)\cdot(E_{A_k}(\Xi')\otimes I)(\psi_2 \otimes \phi_0)\rangle_{V\otimes U}$$
$$= \langle \psi_1 \otimes \phi_0, (E_{A_k}(\Xi')\otimes I)\cdot E_{\hat{A}_k}(\Xi)(\psi_2 \otimes \phi_0)\rangle_{V\otimes U} = \langle \psi_1, E_{A_k}(\Xi')F_k(\Xi)\psi_2\rangle_V,$$

which implies that E_{A_k} and F_k commute. Also, we see that, for any ψ in $\mathrm{Dom}(A_k)$,

$$\int_{\mathbf{R}} \lambda\langle \psi, F_k(\mathrm{d}\lambda)\psi\rangle_V = \int_{\mathbf{R}} \lambda\langle \psi \otimes \phi_0, E_{\hat{A}_k}(\mathrm{d}\lambda)(\psi \otimes \phi_0)\rangle_{V\otimes U}$$
$$= \langle \psi \otimes \phi_0, \hat{A}_k(\psi \otimes \phi_0)\rangle_{V\otimes U} = \langle \psi, A_k\psi\rangle_V = \int_{\mathbf{R}} \lambda\langle \psi, E_{A_k}(\mathrm{d}\lambda)\psi\rangle_V.$$

Therefore, we get (i). Next we see that, for any $\psi \in \mathrm{Dom}(A_k)$,

$$\|\, (\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)\, \|_{V\otimes U}$$
$$= \left[\iint_{\mathbf{R}^2} (\lambda_1 - \lambda_2)^2 \langle \psi \otimes \phi_0, E_{\hat{A}_k}(\mathrm{d}\lambda_1)\cdot(E_{A_k}(\mathrm{d}\lambda_2)\otimes I)(\psi \otimes \phi_0)\rangle_{V\otimes U}\right]^{1/2}$$
$$= \left[\iint_{\mathbf{R}^2} (\lambda_1 - \lambda_2)^2 \langle \psi, F_k(\mathrm{d}\lambda_1)E_{A_k}(\mathrm{d}\lambda_2)\psi\rangle_V\right]^{1/2} = \Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k], S(|\psi\rangle\langle\psi|)\right)\right). \tag{7.2}$$

This completes the proof. □

Remark 7.3. Put $\hat{\mathbf{O}}_k = (\mathbf{R}, \mathscr{B}_{\mathbf{R}}, E_{\hat{A}_k})$, $k = 1, 2$. Then, the above (7.2) implies that

$$\|(\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)\, \|_{V\otimes U} = \Delta\left(\mathbf{M}_{B(V\otimes U)}\left(\hat{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k \otimes I], S(|\psi \otimes \phi_0\rangle\langle\psi \otimes \phi_0|)\right)\right). \tag{7.3}$$

Thus, if we consider Heisenberg's uncertainty relation in the tensor Hilbert space $V \otimes U$, this (7.3) has already clarified the meaning of $\|\, (\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)\, \|_{V\otimes U}$. Therefore, Theorem 7.5 (stated later) is another answer without a tensor Hilbert space.

Lemma 7.4. *Let* A_1 *and* A_2 *be self-adjoint operators in a Hilbert space* V. *Let* $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_{12} \equiv (\mathbf{R}^2, \mathscr{B}_{\mathbf{R}^2}, F \equiv F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2), S(*))$ *be an approximate simultaneous measurement of* A_1 *and* A_2 *in the sense of Definition* 7.1. *Let* $[\hat{F}, U, \phi_0]$ *be any Naimark–Holevo extension of* $\bar{\mathbf{O}}_{12}$, *that is, it satisfies the following conditions* (a) *and* (b): (a) $(\mathbf{R}^2, \mathscr{B}_{\mathbf{R}^2}, \hat{F})$ *is a crisp* W^**-observable in* $B(V \otimes U)$ *and* $\phi_0 \in U$ *such that* $\|\phi_0\|_U = 1$, (b) *it holds that* $\langle \psi_1, F(\Xi_1 \times \Xi_2)\psi_2\rangle_V = \langle \psi_1 \otimes \phi_0, \hat{F}(\Xi_1 \times \Xi_2)(\psi_2 \otimes \phi_0)\rangle_{V\otimes U}$ $(\forall\psi_1, \forall\psi_2 \in V,\ \forall\Xi_1 \in \mathscr{B}_{\mathbf{R}},\ \forall\Xi_2 \in \mathscr{B}_{\mathbf{R}})$. *(The existence of* $[\hat{F}, U, \phi_0]$ *is guaranteed by Naimark–Holevo theorem; cf.* [5]). *Put* $E_{\hat{A}_1}(\Xi) = \hat{F}(\Xi \times \mathbf{R})$, $E_{\hat{A}_2}(\Xi) = \hat{F}(\mathbf{R} \times \Xi)$ *and* $\hat{A}_k = \int_{\mathbf{R}} \lambda E_{\hat{A}_k}(\mathrm{d}\lambda)$. *Then, we see*

(i) *the quartet* $[\hat{A}_1, \hat{A}_2, U, \phi_0]$ *is an approximate simultaneous measurement of* A_1 *and* A_2 *in the sense of Proposition* 1.1.

(ii) $\Delta(\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k], S(|\psi\rangle\langle\psi|))) = \|\, (\hat{A}_k - A_k \otimes I)(\psi \otimes \phi_0)\, \|_{V\otimes U}$ *for all* $\psi \in \mathrm{Dom}(\mathrm{A_k})$.

Proof. By the same way in the proof of Lemma 7.2, we can easily get the proof.

Now we can propose the mathematical foundation of Heisenberg's uncertainty relation. The reader should remark the analogy between the measurement error of pencil's length (Example 4.13) and that of Heisenberg's uncertainty relation.

Theorem 7.5 (Heisenberg's uncertainty relation). *Let A_1 and A_2 be physical quantities (i.e., self-adjoint operators) in a Hilbert space V. Then, the followings hold:*

(A) *There exists an approximate simultaneous measurement $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_{12}, S(*))$ of A_1 and A_2. Furthermore, for any positive ε, we can take this $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_{12} \equiv (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, F \equiv F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2), S(*))$ such that*

$$\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k], S(\bar{\rho})\right)\right) = \begin{cases} \varepsilon(\mathrm{tr}[\bar{\rho}A_1^2]_V)^{1/2} & \forall \bar{\rho} \in \mathrm{Dom}(A_1) \cap \mathrm{Tr}_{+1}(V) \quad \text{if } k = 1, \\ \varepsilon^{-1}(\mathrm{tr}[\bar{\rho}A_2^2]_V)^{1/2} & \forall \bar{\rho} \in \mathrm{Dom}(A_2) \cap \mathrm{Tr}_{+1}(V) \quad \text{if } k = 2, \end{cases}$$

where "$\bar{\rho} \in \mathrm{Dom}(A)$" implies that $\int_{\mathbf{R}} \lambda^2 \, \mathrm{tr}[\bar{\rho} \cdot E_A(\mathrm{d}\lambda)]_V < \infty$.

(B) *For any approximate simultaneous measurement $\mathbf{M}_{B(V)}(\bar{\mathbf{O}}_{12} \equiv (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, F \equiv F_1 \overset{\bar{\mathbf{O}}_{12}}{\times} F_2), S(*))$ of A_1 and A_2 and for any $\bar{\rho} \in \mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2) \cap \mathrm{Tr}_{+1}(V)$, the following inequality (Heisenberg's uncertainty relation) holds:*

$$\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_1 \overset{\mathbf{u}}{\times} [A_1], S(\bar{\rho})\right)\right) \cdot \Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_2 \overset{\mathbf{u}}{\times} [A_2], S(\bar{\rho})\right)\right) \geqslant |\mathrm{tr}[\bar{\rho} \cdot (A_1A_2 - A_2A_1)]_V|/2.$$

Proof. First we assume that $\bar{\rho}$ is a pure state, i.e., $\bar{\rho} = |\psi\rangle\langle\psi|$. Then, the statement (A) immediately follows from Proposition 1.1 (A) and Lemma 7.2. Also, we can easily see, by Lemma 7.4 and Proposition 1.1 (B), that the statement (B) holds since it holds that $\mathrm{tr}[\bar{\rho} \cdot (A_1A_2 - A_2A_1)]_V = \langle A_1\psi, A_2\psi\rangle_V - \langle A_2\psi, A_1\psi\rangle_V$.

Next we consider the general case that $\bar{\rho} = \sum_{\lambda\in\lambda} \alpha_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|$, where $\{\psi_\lambda | \lambda \in \Lambda\}$ is a complete orthonormal basis in V and $\{a_\lambda\}_{\lambda\in\Lambda}$ is a sequence of non-negative numbers such that $\sum_{\lambda\in\Lambda} a_\lambda = 1$. Note that

$$\begin{aligned} \Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k], S(\bar{\rho})\right)\right) &= \left[\iint_{\mathbf{R}^2} |x_k - \lambda|^2 \mathrm{tr}[\bar{\rho} F_k(\mathrm{d}x_k) E_{A_k}(\mathrm{d}\lambda)]_V\right]^{1/2} \\ &= \left[\sum_{\lambda\in\Lambda} \alpha_\lambda \iint_{\mathbf{R}^2} |x_k - \lambda|^2 \mathrm{tr}[(|\psi_\lambda\rangle\langle\psi_\lambda|) \cdot F_k(\mathrm{d}x_k) E_{A_k}(\mathrm{d}\lambda)]_V\right]^{1/2} \\ &= \left[\sum_{\lambda\in\Lambda} \alpha_\lambda \left|\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_k \overset{\mathbf{u}}{\times} [A_k], S(|\psi_\lambda\rangle\langle\psi_\lambda|)\right)\right)\right|^2\right]^{1/2}. \end{aligned} \tag{7.4}$$

Therefore, from this and the statement (A) for $|\psi_\lambda\rangle\langle\psi_\lambda|$, we see, for example (i.e., $k = 1$), that, for any $\bar{\rho} \in \mathrm{Dom}(A_1)$, i.e., $\bar{\rho} = \sum_{\lambda\in\Lambda} \alpha_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|$ such that $\sum_{\lambda\in\Lambda} \alpha_\lambda ||A_1\psi_\lambda||_V^2 = \int_{\mathbf{R}} \lambda^2 \mathrm{tr}[\bar{\rho} \cdot E_{A_1}(\mathrm{d}\lambda)]_V < \infty$,

$$\begin{aligned} \left[\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_1 \overset{\mathbf{u}}{\times} [A_1], S(\bar{\rho})\right)\right)\right]^2 &= \sum_{\lambda\in\Lambda} \alpha_\lambda \varepsilon^2 ||A_1\psi_\lambda||_V^2 = \sum_{\lambda\in\Lambda} \alpha_\lambda \varepsilon^2 \mathrm{tr}[(|\psi_\lambda\rangle\langle\psi_\lambda|) A_1^2]_V \\ &= \varepsilon^2 \mathrm{tr}\left[\left(\sum_{\lambda\in\Lambda} \alpha_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|\right) A_1^2\right]_V = \varepsilon^2 \mathrm{tr}[\bar{\rho} A_1^2]_V. \end{aligned}$$

Thus this completes the proof of (A) for mixed states. Also, from (7.4) and the statement (B) for $|\psi_\lambda\rangle\langle\psi_\lambda|$, we see that

$$
\begin{aligned}
|\mathrm{tr}[\bar{\rho}\cdot(A_1A_2 - A_2A_1)]_V|/2 &= \left|\sum_{\lambda\in\Lambda}\alpha_\lambda(\langle A_1\psi_\lambda, A_2\psi_\lambda\rangle_V - \langle A_2\psi_\lambda, A_1\psi_\lambda\rangle_V)\right| \Big/ 2 \\
&\leqslant \sum_{\lambda\in\Lambda}\alpha_\lambda|\langle A_1\psi_\lambda, A_2\psi_\lambda\rangle_V - \langle A_2\psi_\lambda, A_1\psi_\lambda\rangle_V|/2 \\
&\leqslant \sum_{\lambda\in\Lambda}\alpha_\lambda \Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_1 \overset{\mathbf{u}}{\times}[A_1], S(|\psi_\lambda\rangle\langle\psi_\lambda|)\right)\right)\cdot\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_2 \overset{\mathbf{u}}{\times}[A_2], S(|\psi_\lambda\rangle\langle\psi_\lambda|)\right)\right) \\
&\leqslant \left[\sum_{\lambda\in\Lambda}\alpha_\lambda\left|\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_1 \overset{\mathbf{u}}{\times}[A_1], S(|\psi_\lambda\rangle\langle\psi_\lambda|)\right)\right)\right|^2\right]^{1/2} \\
&\quad\cdot\left[\sum_{\lambda\in\Lambda}\alpha_\lambda\left|\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_2 \overset{\mathbf{u}}{\times}[A_2], S(|\psi_\lambda\rangle\langle\psi_\lambda|)\right)\right)\right|^2\right]^{1/2} \quad \text{(by Schwarz inequality)} \\
&= \Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_1 \overset{\mathbf{u}}{\times}[A_1], S(\bar{\rho})\right)\right)\cdot\Delta\left(\mathbf{M}_{B(V)}\left(\bar{\mathbf{O}}_2 \overset{\mathbf{u}}{\times}[A_2], S(\bar{\rho})\right)\right).
\end{aligned}
$$

This completes the proof. □

8. Conclusions

In this paper, we proposed "subjective fuzzy measurement theory". This theory was characterized as the subjective (or, statistical) method of "objective fuzzy measurement theory". The original motivation was to define "measurement error" in a general situation. In fact, in order to appreciate Example 4.13 (the measurement error of pencil's length), we must understand almost contents in this paper except the arguments (i.e., fuzzy sets theory) in Section 6. And, under these preparations, we clarified Heisenberg's uncertainty relation.

We think that our motivation was proper. That is because "subjective fuzzy measurement theory" is a general fuzzy theory. Other fuzzy theories (except fuzzy sets theory) are characterized as some aspects of subjective fuzzy measurement theory (cf. Remark 6.5). Also we can assert the objective aspect of fuzzy sets theory, that is, it belongs to "objective fuzzy measurement theory". However, the most important assertion is not "generality" but "objectivity". That is, all results in this paper are consequences of Axioms 0 and 1 (i.e., objective fuzzy measurement theory). In this sense, Theorem 3.6 is essential (cf. Remark 4.8).

We believe that our observation is merely the first step in fuzzy measurement theory. Thus we expect the developments of this theory.

Note added in proof

For the further arguments concerning fuzzy sets theory, see my paper "Fuzzy logic in measurements" (submitted to Fuzzy sets and Systems). The preprint can be found on my homepage [http://www.math.keio.ac.jp/~ishikawa].

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