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Fuzzy inferences by algebraic method

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Abstract

In this paper we propose a foundation of fuzzy measurement theory, which is described in terms of C^*-algebras. This theory is a general measurement theory for classical and quantum systems. Here we also propose the identification: "measurement" = "inference". As a "fuzzy" aspect of our proposal, we show and prove several fuzzy syllogisms in classical systems. Since our theory is concerning measurements (and not mathematics), these syllogisms should be regarded as objective facts in our real world. That is, our theory is "objective fuzzy theory". Therefore, we believe that our proposal is a straight approach to "fuzziness". © 1997 Elsevier Science B.V.

Keywords: C^*-algebra; Quantum mechanics; Fuzzy observable; Measurement; Fuzzy inference; Fuzzy syllogism

1. Introduction

In this paper we propose a foundation of fuzzy measurement theory (or in short, measurement theory), which is a general measurement theory for classical and quantum systems.

In Section 2 we propose an algebraic formulation (i.e., axiom) of measurement theory. There we also propose the identification: "measurement" = "inference", that is, their mathematical representations are the same. In Section 3 we study the simple properties of quasi-product observables and introduce "implication" in measurement theory. As a "fuzzy" aspect of our axiom, in Section 4 we show several theorems of fuzzy syllogisms in classical systems. For example, under the condition that "$A \Rightarrow B$, $B \Rightarrow C$", we can assert a kind of conclusion such as "$C \Rightarrow A$". It is also possible to assert the justification of the standard syllogism (i.e., "$A \Rightarrow B$, $B \Rightarrow C$" implies "$A \Rightarrow C$") for classical systems. This result is never trivial but rather remarkable because logic can assert nothing for the justification. Hence, in the light of our theory, for the first time we can understand actual syllogisms for classical systems. It should be noted that our proposal in Section 2 is not a rule in mathematics but a principle that dominates all measurements in science. Therefore, all results derived from our axiom are objective facts in our real world if our axiom is true. That is, our proposal is an "objective fuzzy theory". This is most desirable because any theory (except "mathematics" or "method") should be objective, if possible. Therefore, we believe that our proposal is a straight approach to "fuzziness".

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2. Algebraic formulation of fuzzy measurement theory

In this section we propose an algebraic foundation of fuzzy measurement theory (i.e., measurement theory). We believe that "fuzziness" (precisely, "objective fuzziness" proposed in this paper) is one of the most fundamental concepts in science. And thus, the mathematical preparations are a little difficult. C^*-algebras are indispensable for a foundation of measurement theory (just like differential equations are indispensable for classical mechanics). We do not impose on the readers the need for a mathematical knowledge of C^*-algebras. We therefore begin with the definition of the C^*-algebra (cf. [11]).

Let $\mathscr{A}$ be a linear associative algebra over the complex field $\mathbb{C}$. The algebra $\mathscr{A}$ is called a *Banach algebra* if it is associated to each element T a real number $\|T\|$, called the *norm* of T, with the properties:

(i) $\|T\| \geqslant 0$,
(ii) $\|T\| = 0$ if and only if $T = 0$ (i.e., the 0-element in $\mathscr{A}$),
(iii) $\|T + S\| \leqslant \|T\| + \|S\|$,
(iv) $\|\lambda T\| = |\lambda| \cdot \|T\|$, $\lambda \in \mathbb{C}$.
(v) $\|TS\| \leqslant \|T\| \cdot \|S\|$,
(vi) $\mathscr{A}$ is complete with respect to the norm $\| \cdot \|$.

A mapping $T \mapsto T^*$ of $\mathscr{A}$ into itself is called an *involution* if it satisfies the following conditions:

(i) $(T^*)^* = T$,
(ii) $(T + S)^* = T^* + S^*$,
(iii) $(TS)^* = S^*T^*$,
(iv) $(\lambda T)^* = \bar{\lambda} T^*$, $\lambda \in \mathbb{C}$.

A Banach algebra with an involution * is called a *Banach*-algebra*.

Definition 2.1. A Banach*-algebra $\mathscr{A}$ is called a *C^*-algebra* if it satisfies $\|T^*T\| = \|T\|^2$ for any $T \in \mathscr{A}$. In this paper we always suppose that a *C^*-algebra $\mathscr{A}$* has the identity element I (i.e., $IT = TI = T$ for all $T \in \mathscr{A}$). Also, a C^*-algebra $\mathscr{A}$ is called commutative if $T_1T_2 = T_2T_1$ ($\forall T_1, T_2 \in \mathscr{A}$).

The following are some typical examples of C^*-algebras, which will clarify its nature to those who are not yet familiar with the notion.

Example 2.2. (i) Let Ω be a compact Hausdorff space, and $C(\Omega)$ be the algebra, under pointwise multiplication, of all complex valued, continuous functions on Ω. Define $\|f\| = \max_{\omega\in\Omega} |f(\omega)|$, and $f^*(\omega) = \overline{f(\omega)}$ ($\forall \omega \in \Omega$). Then, $C(\Omega)$ is a commutative C^*-algebra, that is, $f_1f_2 = f_2f_1$ holds for any f_1, f_2 ($\in C(\Omega)$). Of course, f_I(resp. f_0) defined by $f_I(\omega) = 1$ (resp. $f_I(\omega) = 0$), $\forall \omega \in \Omega$, is the identity element (resp. 0-element) in $C(\Omega)$. Also, we need to recall Gelfand theorem: Any commutative C^*-algebra $\mathscr{A}$ *(with the identity) is *-isomorphic to the C^*-algebra* $C(\Omega)$. That is, we can always identify a commutative C^*-algebra $\mathscr{A}$ with $C(\Omega)$.

(ii) Let V be a Hilbert space. Let

$$B(V) = \{T : T \text{ is a bounded linear operator from a Hilbert space } V \text{ into itself }\}.$$

Define $\|T\|_{B(V)} = \sup\{\|Tv\|_V : \|v\|_V = 1\}$, and $(T_1T_2)(v) = T_1(T_2v)$ ($\forall v \in V$). Let T^* be the adjoint operator of T. This $B(V)$ is of course a non-commutative C^*-algebra in general. Also note that $\mathscr{C}_c(V) \equiv \{T \in B(V) : T$ is a compact operator$\}$ is a C^*-subalgebra of $B(V)$. If the dimension of V is infinite, $\mathscr{C}_c(V)$ has no identity I. Thus we define $\mathscr{C}(V)$ as the smallest C^*-algebra that contains I and $\mathscr{C}_c(V)$.

The spectrum $\mathrm{Sp}(T)$ of an element T in a C^*-algebra $\mathscr{A}$ is defined by the set $\{\lambda \in \mathbb{C}$: $(T - \lambda I)^{-1}$ does not exist$\}$; that is, $\lambda \in \mathrm{Sp}(T)$ if and only if there exists no element S in $\mathscr{A}$ such that $S(T - \lambda I) = (T - \lambda I)S = I$. Note, for example, that $\mathrm{Sp}(T)$ is the set of all eigenvalues of T if $T \in B(\mathbb{C}^n)$ (i.e., $V = \mathbb{C}^n$ in Example 2.2(ii)). Also, if $f \in C(\Omega)$ in Example 2.2(i), then $\mathrm{Sp}(f) = \{f(\omega) : \omega \in \Omega\}$, that is, the range of f. In general, $\mathrm{Sp}(T)$

is a closed subset in the complex field $\mathbb{C}$. An element T in C^*-algebra $\mathscr{A}$ is called *self-adjoint* if $T = T^*$. A self-adjoint element T is called *positive* (resp. *projection*) if $\mathrm{Sp}(T) \subseteq [0,\infty)$ (resp. $\mathrm{Sp}(T) \subseteq \{0,1\}$ (or equivalently, $T = T^2$)). A positive element T is sometimes denoted by $T \geqslant 0$.

Note that a C^*-algebra $\mathscr{A}$ is also a Banach space. So, $\mathscr{A}$ has the dual Banach space $\mathscr{A}^* \equiv \{\rho : \rho$ is a continuous linear functional on $\mathscr{A}\}$ with the norm $\|\cdot\|_{\mathscr{A}^*}$ (i.e., $\|\rho\|_{\mathscr{A}^*} \equiv \sup\{|\rho(T)| : \|T\|_{\mathscr{A}} \leqslant 1\}$). Define the *mixed-state class* $\mathfrak{S}(\mathscr{A}^*)$ such as $\mathfrak{S}(\mathscr{A}^*) \equiv \{\rho \in \mathscr{A}^* : \|\rho\|_{\mathscr{A}^*} = 1$ and $\rho(T^*T) \geqslant 0$ for all $T \in \mathscr{A}\}$. A mixed state ρ^p (i.e., $\rho^p \in \mathfrak{S}(\mathscr{A}^*)$) is called a *pure state*, if "$\rho^p = \lambda\rho_1 + (1-\lambda)\rho_2$ ($\rho_1, \rho_2 \in \mathfrak{S}(\mathscr{A}^*)$, and $0 < \lambda < 1$)" implies "$\rho^p = \rho_1 = \rho_2$". Define

$$\mathfrak{S}^p(\mathscr{A}^*) \equiv \{\rho^p \in \mathfrak{S}(\mathscr{A}^*) : \rho^p \text{ is a pure state}\}.$$

Example 2.3. The following identifications are well-known in functional analysis (cf. [13]):

$$C(\Omega)^* \approx \mathscr{M}(\Omega) \text{ (as Banach space) and } \mathfrak{S}(C(\Omega)^*) \approx \mathscr{M}_{+1}(\Omega),$$

where $C(\Omega)^*$ is the dual Banach space of $C(\Omega)$ and

$$\mathscr{M}(\Omega) = \{\mu : \mu \text{ is a regular signed measure on } \Omega\},$$

$$\mathscr{M}_{+1}(\Omega) = \{\mu \in \mathscr{M}(\Omega) : \mu \text{ is non-negative and } \mu(\Omega) = 1\}.$$

Also we see

$$\mathfrak{S}^p(C(\Omega)^*) \approx \mathscr{M}^p_{+1}(\Omega) = \{\delta_{\omega_0} \in \mathscr{M}_{+1}(\Omega) : \omega_0 \in \Omega\},$$

where δ_{ω_0} is a point measure at ω_0, i.e., $\delta_{\omega_0}(f) = \int_\Omega f(\omega)\delta_{\omega_0}(\mathrm{d}\omega) = f(\omega_0)$ $(\forall f \in C(\Omega))$.

The concept of "fuzzy observable" was first introduced in quantum mechanics by Davies [3]. (Also see [5]). The following definition (for fuzzy systems) is an easy generalization of his idea. "Observable" in physics naturally represents "physical quantity", i.e., position, momentum, energy, etc.

Definition 2.4 (*Fuzzy observable, crisp observable*). Let $\mathscr{A}$ be a C^*-algebra. A *fuzzy observable* (or in short, *observable*) $(X, \mathscr{P}(X), F)$ in $\mathscr{A}$ is defined such that

(i) a label set X is a finite set, that is,

$$X = \{x^1, x^2, \ldots, x^J\}$$

and $\mathscr{P}(X)$ is the power set of X, i.e., $\mathscr{P}(X) = \{\Xi : \Xi \subseteq X\}$,

(ii) for every $\Xi \in \mathscr{P}(X)$, $F(\Xi)$ is a positive element in $\mathscr{A}$ such that $F(\emptyset) = 0$ and $F(X) = I$, where 0 is the 0-element and I is the identity element in $\mathscr{A}$ and

(iii) the following holds:

$$F(\Xi) = \sum_{x \in \Xi} F(\{x\}) \quad (\forall \Xi \in \mathscr{P}(X)). \tag{2.1}$$

Also, if $F(\Xi)$ is a projection for every Ξ $(\in \mathscr{P}(X))$, a fuzzy observable $(X, \mathscr{P}(X), F)$ is called a *crisp observable*.

Remark 2.5. Let $(X, \mathscr{P}(X), F)$ be an observable in $\mathscr{A}$ and let $\rho \in \mathfrak{S}(\mathscr{A}^*)$. Then, putting $\mu(\Xi) = \rho(F(\Xi))$ $(\forall \Xi \in \mathscr{P}(X))$, we see that $(X, \mathscr{P}(X), \mu)$ is a probability space.

Let $(Z, \mathscr{P}(Z), H)$ be an observable in a C^*-algebra $\mathscr{A}$. Let g be a map from Z into Y. Then, we can define the observable $(Y, \mathscr{P}(Y), G)$ in $\mathscr{A}$ such that $G(\Gamma) = H(g^{-1}(\Gamma))$ $(\forall \Gamma \in \mathscr{P}(Y))$. This observable $(Y, \mathscr{P}(Y), G)$ is called an *image observable of* g for $(Z, \mathscr{P}(Z), H)$.

Let K be a finite set, that is, $K = \{1,2,\ldots,|K|\}$. (We do not regard this K as an ordered set.) For each $k \in K$, consider a finite set $X_k = \{x_k^1, x_k^2, \ldots, x_k^{J_k}\}$. The product set of X_k's is defined by

$$\mathop{\times}_{k\in K} X_k = \{(x_k)_{k\in K} = (x_1,x_2,\ldots,x_{|K|}) : x_1 \in X_1, x_2 \in X_2, \ldots, x_{|K|} \in X_{|K|}\}.$$

Consider Ξ_k $(\in \mathscr{P}(X_k))$ for each $k \in K$. Define $\times_{k\in K}\Xi_k$ by $\{(x_k)_{k\in K} : x_k \in \Xi_k,\ k \in K\}$. Since K is not regarded as an ordered set, we can write, for example, $\times_{k\in K}\Xi_k = (\times_{k\in D}\Xi_k) \times (\times_{k\in K\setminus D}\Xi_k) = (\times_{k\in K\setminus D}\Xi_k) \times (\times_{k\in D}\Xi_k)$, where $D \subseteq K$.

Definition 2.6 (*Marginal observable, quasi-product observable, consistency*). Let $\mathscr{A}$ be a C^*-algebra. Let $K = \{1,2,\ldots,|K|\}$.

(i) Consider an observable $\mathbf{O} \equiv (\times_{k\in K} X_k, \mathscr{P}(\times_{k\in K} X_k), F)$ (with a label set $\times_{k\in K} X_k$) in $\mathscr{A}$. Let $D \subseteq K$. An observable $\mathbf{O}_D \equiv (\times_{k\in D} X_k, \mathscr{P}(\times_{k\in D} X_k), F_D)$ in $\mathscr{A}$ is called a *D-marginal observable* of $\mathbf{O}$ if it satisfies

$$F_D\left(\mathop{\times}_{k\in D} \Xi_k\right) = F\left(\left(\mathop{\times}_{k\in D} \Xi_k\right) \times \left(\mathop{\times}_{k\in K\setminus D} X_k\right)\right)$$

for all $\Xi_k \in \mathscr{P}(X_k)$, $k \in D$. Also this $\mathbf{O}_D$ is denoted by $\mathbf{O}|_D$. Here note that the marginal observable $\mathbf{O}|_D$ is equal to the image observable of g_D for $\mathbf{O}$, where $\times_{k\in K} X_k \ni (x_k)_{k\in K} \xmapsto{g_D} (x_k)_{k\in D} \in \times_{k\in D} X_k$.

(ii) For each $k \in K$, consider an observable $\mathbf{O}_k \equiv (X_k, \mathscr{P}(X_k), F_k)$ in $\mathscr{A}$. If there exists an observable $\mathbf{O}_K \equiv (\times_{k\in K} X_k, \mathscr{P}(\times_{k\in K} X_k), F)$ in $\mathscr{A}$ such that $\mathbf{O}_K|_{\{k\}} = \mathbf{O}_k$ for all $k \in K$, then $[\mathbf{O}_k : k \in K]$ is called *consistent*. Also, this $\mathbf{O}_K$ is called a *quasi-product observable of* $[\mathbf{O}_k : k \in K]$, and is sometimes denoted by $(\times_{k\in K} X_k, \mathscr{P}(\times_{k\in K} X_k), \times_{k\in K}^{\mathbf{O}_K} F_k)$, or $\times_{k\in K}^{\mathbf{O}_K} \mathbf{O}_k$.

Note that the consistency of observables $[(X_k, \mathscr{P}(X_k), F_k) : k \in K]$ in $\mathscr{A}$ is not guaranteed in general. The following lemma is well-known (and easy).

Lemma 2.7. *If the commutativity condition*

$$F_{k_1}(\Xi_{k_1})F_{k_2}(\Xi_{k_2}) = F_{k_2}(\Xi_{k_2})F_{k_1}(\Xi_{k_1}) \quad (\forall \Xi_{k_1} \in \mathscr{P}(X_{k_1}),\ \forall \Xi_{k_2} \in \mathscr{P}(X_{k_2}),\ k_1 \neq k_2)$$

holds, then we can construct a quasi-product observable $\mathbf{O} \equiv (\times_{k\in K} X_k, \mathscr{P}(\times_{k\in K} X_k), F \equiv \times_{k\in K}^{\mathbf{O}} F_k)$ *such that*

$$F(\Xi_1 \times \Xi_2 \times \ldots \times \Xi_{|K|}) = F_1(\Xi_1)F_2(\Xi_2)\ldots F_{|K|}(\Xi_{|K|}).$$

It is, of course, the case that the uniqueness is not guaranteed even under the above commutativity condition (see Lemma 3.1).

Based on the above mentioned mathematical preparations, it is viable to propose a foundation of fuzzy measurement theory. Since this theory is concerning measurements (and not mathematics), for each fundamental object in measurements, it is essential to determine its mathematical representation. If we have no procedure of this kind, our theory will be "mathematics" or "subjective theory" (i.e., "method"). But that is not our intention. The procedure will be done by the analogy of quantum mechanics. That is because physics is a typical objective theory. As the most basic requirement for a fuzzy theoretical description of a fuzzy system we have the following axiom:

Axiom 0 (*Fuzzy system, state, observable, measurement, measured value, true value*). With any *fuzzy system* (or in short, *system*) S, a C^*-algebra $\mathscr{A}$ can be associated in which the fuzzy measurement theory of that system can be formulated.

(i) A state Θ of the fuzzy system S is represented by a pure state ρ^p $(\in \mathfrak{S}^p(\mathscr{A}^*))$; and an *observable* $\mathcal{O}_Z$ (with a label set Z) is represented by an observable (defined in Definition 2.4) $\mathbf{O} = (Z, \mathscr{P}(Z), H)$ in the C^*-algebra $\mathscr{A}$. Also, the *measurement* $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$, i.e., the measurement of the observable $\mathcal{O}_Z$ for the system S with the state Θ, is represented by $\mathbf{M}(\mathbf{O}, S_{\rho^p})$ in the C^*-algebra $\mathscr{A}$.

(ii) We can get a *measured value* z $(\in Z)$ by the measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$.

(iii) Let $\mathcal{O}_Y$ be an observable, which is represented by the image observable $(Y, \mathscr{P}(Y), G)$ of $g : Z \to Y$ for $\mathbf{O}$. (Here $\mathcal{O}_Y$ is also called an *image observable* of g for $\mathcal{O}_Z$.) When we get the measured value z by the measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$, we consider that the *value* (or, *true value*) of $\mathcal{O}_Y$ (for the system S with the state Θ) is equal to $g(z)$.

Remark 2.8. A fuzzy system S always has its state Θ (or its mathematical representation ρ^p $(\in \mathfrak{S}^p(\mathscr{A}^*))$). Thus it should be denoted by S_Θ (or S_{ρ^p}). However, we sometimes do not know the state Θ (or ρ^p) of a fuzzy system S. Hence, we sometimes denote $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$ by $\mathscr{M}(\mathcal{O}_Z, S)$ (or $\mathbf{M}(\mathbf{O}, S_{\rho^p})$ by $\mathbf{M}(\mathbf{O}, S)$).

Another axiom presented below is analogous to (or, a kind of generalizations of) Born's probabilistic interpretation of quantum mechanics.

Axiom 1 (*Probabilistic interpretation of measurement*). Consider a measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$, which is represented by $\mathbf{M}(\mathbf{O} \equiv (Z, \mathscr{P}(Z), H), S_{\rho^p})$ in a C^*-algebra $\mathscr{A}$. Assume that z $(\in Z)$ is the measured value obtained by the measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$. Then,

(*) the probability that the $z (\in Z)$ belongs to a set Ξ $(\in \mathscr{P}(Z))$ is given by $\rho^p(H(\Xi))$.

Our main proposals are the two axioms (i.e., Axioms 0 and 1) stated above. We often identify $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$ with $\mathbf{M}(\mathbf{O}, S_{\rho^p})$. The measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$ (or, its mathematical representation $\mathbf{M}(\mathbf{O}, S_{\rho^p})$) is sometimes called a *simultaneous measurement* if $\mathcal{O}_Z$ (or, its mathematical representation $\mathbf{O}$) is a quasi-product observable, i.e., the label set Z is considered as the product set $\times_{k \in K} X_k$. (Here the word "simultaneous" is independent of time.) Putting $Z = \times_{k\in K} X_k$, $Y = \times_{k \in D} X_k$ (where $D \subseteq K$) and $g = g_D$ such that $\times_{k \in K} X_k \ni (x_k)_{k\in K} \overset{g_D}{\longmapsto} (x_k)_{k \in D} \in \times_{k\in D} X_k$, we get the following axiom as a direct consequence of Axioms 0 and 1.

Axiom 1′ (*Simultaneous measurements*). Consider a simultaneous measurement $\mathscr{M}(\mathcal{O}_{\times_{k\in K} X_k}, S_\Theta)$, which is represented by $\mathbf{M}(\mathbf{O} \equiv (\times_{k\in K} X_k, \mathscr{P}(\times_{k\in K} X_k), \times^{\mathbf{O}}_{k\in K} F_k), S_{\rho^p})$ in a C^*-algebra $\mathscr{A}$. Let $D \subseteq K$. (Thus the D-marginal observable $\mathbf{O}|_D \equiv (\times_{k\in D} X_k, \mathscr{P}(\times_{k\in D} X_k), F_D)$ in $\mathscr{A}$ represents an observable $\mathcal{O}_{\times_{k\in D} X_k}$, which is also called the *D-marginal observable* of $\mathcal{O}_{\times_{k\in K} X_k}$.) Assume that $x (= (x_k)_{k\in K} \in \times_{k\in K} X_k)$ is a measured value obtained by the measurement $\mathscr{M}(\mathcal{O}_{\times_{k\in K} X_k}, S_\Theta)$. Then,

(i) the probability that the x $(= (x_k)_{k\in K} \in \times_{k\in K} X_k)$ belongs to a set Ξ $(\in \mathscr{P}(\times_{k\in K} X_k))$ is given by $\rho^p((\times^{\mathbf{O}}_{k\in K} F_k)(\Xi))$,

(ii) the $(x_k)_{k\in D}$ $(= g_D(x) = g_D((x_k)_{k\in K}))$ is regarded as the value of the observable $\mathcal{O}_{\times_{k\in D} X_k}$ for the system S_Θ (obtained by this measurement $\mathscr{M}(\mathcal{O}_{\times_{k\in K} X_k}, S_\Theta)$).

In particular, putting $D = \{k\}$ $(\forall k \in K)$ in (ii) above, we see that

(iii) the x_k $(= g_{\{k\}}((x_k)_{k\in K}) \in X_k)$ can be regarded as the value of the observable $\mathcal{O}_{X_k}$ for the system S_Θ.

Now we investigate "fuzzy inference" (or in short, "inference"). We propose the identification: "inference"= "measurement". (We will show that, throughout this paper this identification is justified). That is, "fuzzy inference" is another form of "fuzzy measurement" (i.e. "measurement"). The only difference between the two is that of the view-points. Therefore, the following Axiom 1″ is another form of Axiom 1′ from a different point of view. For simplicity, we identify $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$ with $\mathbf{M}(\mathbf{O}, S_{\rho^p})$ in the following axiom (and in what follows). We hope that the reader does not confuse a measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$ with its mathematical representation $\mathbf{M}(\mathbf{O}, S_{\rho^p})$. It is obvious that we can take an actual measurement $\mathscr{M}(\mathcal{O}_Z, S_\Theta)$ even if we do not know $\mathbf{M}(\mathbf{O}, S_{\rho^p})$.

Here we can state the following axiom as another form of Axiom 1′ for $K = \{1,2\}$ (or, replacing $\mathsf{X}_{k\in D}X_k$ (resp. $\mathsf{X}_{k\in K\setminus D}X_k$) by X_1 (resp. X_2)).

Axiom 1″ (*Fuzzy inferences*). Let S be a fuzzy system with the pure state ρ^p ($\in \mathfrak{S}^p(\mathscr{A}^*)$), which is formulated in a C^*-algebra $\mathscr{A}$. Let $\mathbf{O} = (X_1 \times X_2, \mathscr{P}(X_1 \mathsf{X} X_2), F_1 \mathsf{X}^{\mathbf{O}} F_2)$ be a quasi-product observable of $(X_1, \mathscr{P}(X_1), F_1)$ and $(X_2, \mathscr{P}(X_2), F_2)$. Then, if we know the value x_1 ($\in X_1$) of the observable $(X_1, \mathscr{P}(X_1), F_1)$ for the system S_{ρ^p}, we can infer, by the fuzzy inference $\mathbf{M}(\mathbf{O}, S_{\rho^p})$, that the probability that x_2 ($\in X_2$), the value of the observable $(X_2, \mathscr{P}(X_2), F_2)$ for the system S_{ρ^p}, belongs to a set Ξ_2 ($\in \mathscr{P}(X_2)$) is given by $P_{\mathbf{M}(\mathbf{O},S_{\rho^p})}(x_1, \Xi_2)$, where

$$P_{\mathbf{M}(\mathbf{O},S_{\rho^p})}(x_1, \Xi_2) = \frac{\rho^p((F_1 \mathsf{X}^{\mathbf{O}} F_2)(\{x_1\} \times \Xi_2))}{\rho^p((F_1 \mathsf{X}^{\mathbf{O}} F_2)(\{x_1\} \times X_2))}. \tag{2.2}$$

Here we assume, for convenience, that this expression is equal to $|\Xi_2|/|X_2|$ if $\rho^p((F_1 \mathsf{X}^{\mathbf{O}} F_2)(\{x_1\} \times X_2)) = 0$.

Remark 2.9. For simplicity, we omitted several arguments: for example, the arguments about the case that a label set X is infinite, a C^*-algebra $\mathscr{A}$ has no identity, and so on. For further arguments, see [8].

Remark 2.10. Note that a quantum system S is a kind of fuzzy system, which is described in a non-commutative C^*-algebra $\mathscr{C}(V)$ (cf. Example 2.2 (ii)). In other words, our axiom includes Born's probabilistic interpretation of quantum mechanics. Hence, if quantum theory (i.e., Born's axiom) is not true, our proposal is not true either. All results derived from our axiom, as well as Born's axiom, are facts that should be tested by serious experiments. That is because our proposal is not a rule in mathematic but a principle that dominates all measurements in science. In fact, Bell showed in [2] that Born's axiom predicted a surprising fact, i.e., the existence of something faster than light. In spite of this unbelievable prediction, Born's axiom has been well authorized by serious experiments (cf. [1, 4, 12]). We introduce the following classification in fuzzy measurement theory:

$$\text{Fuzzy measurement theory} \begin{cases} \text{commutative fuzzy theory (for classical systems),} \\ \text{Non-commutative fuzzy theory (for quantum systems),} \end{cases}$$

where a C^*-algebra $\mathscr{A}$ is commutative or non-commutative.

Remark 2.11. If we want the data concerning both $\mathbf{O}_1$ and $\mathbf{O}_2$ for the system S_{ρ^p}, according to Axiom 1′(iii) for $K = \{1,2\}$ we must take a simultaneous measurement $\mathbf{M}(\mathbf{O}_{12} \equiv \mathbf{O}_1 \mathsf{X}^{\mathbf{O}_{12}} \mathbf{O}_2, S_{\rho^p})$. Therefore, if a quasi-product observable $\mathbf{O}_{12}$ does not exist (i.e., $[\mathbf{O}_1, \mathbf{O}_2]$ is not consistent), the concept of "the data concerning $\mathbf{O}_1$ and $\mathbf{O}_2$ for the system S_{ρ^p}" is nonsense, i.e., it has no reality. This is a prevalent notion in quantum theory as in the case that the concept "the momentum and position of a particle" or "the trajectory of a particle" is meaningless in quantum theory. (For the recent results, see [6,7,9].) It should be emphasized that the importance of this spirit (i.e., "the consistency of $[\mathbf{O}_1, \mathbf{O}_2]$" $\Leftrightarrow$ "the reailty of data") is essential throughout this paper.

3. Quasi-product observable and implication

Most of the remaining part of this paper will be devoted to investigating classical systems in measurement theory. As the commutative fuzzy measurement theory is rather easy, people tend to overlook important facts in classical systems. Since quantum theory is moderately difficult, it is rather handy compared to classical fuzzy theory. Hence we will investigate classical fuzzy systems in comparison with quantum theory. In this section we study the properties of quasi-product observables and implication in measurement theory, which are the preparations for the next section. Here, again, note that we always identify $\mathscr{M}(\mathscr{O}_Z, S_\Theta)$ with $\mathbf{M}(\mathbf{O}, S_{\rho^p})$.

Let $X = \{x^1, x^2, \dots, x^J\}$. Let $\mathbf{O} \equiv (X, \mathscr{P}(X), F)$ be an observable in a commutative C^*-algebra $\mathscr{A}$ (hence by Gelfand theorem in Example 2.2, we can assume that $\mathscr{A} = C(\Omega)$). From (2.1), we can consider the following identification:

$$(X, \mathscr{P}(X), F) \longleftrightarrow \Big[[F(\{x^j\})](\omega) : j = 1, 2, \dots, J \Big]$$

(where $F(\{x^j\}) \equiv [F(\{x^j\})] \in C(\Omega)$), and therefore denote

$$\mathrm{Rep}[\mathbf{O}] = \mathrm{Rep}[(X, \mathscr{P}(X), F)] = \Big[[F(\{x^j\})](\omega) : j = 1, 2, \dots, J \Big].$$

It is clear that

$$0 \leqslant [F(\{x^j\})](\omega) \leqslant 1 \quad \text{and} \quad \sum_{j=1}^{J} [F(\{x^j\})](\omega) = 1 \qquad (\forall \omega \in \Omega),$$

that is, each $F(\{x^j\})$ is a membership function (cf. [14]) on a compact Hausdorff space Ω.

Consider two observables $\mathbf{O}_1 \equiv (X_1, \mathscr{P}(X_1), F_1)$ and $\mathbf{O}_2 \equiv (X_2, \mathscr{P}(X_2), F_2)$ in $C(\Omega)$ such that

$$X_1 = \{x_1^1, x_1^2, \dots, x_1^{J_1}\} \quad \text{and} \quad X_2 = \{x_2^1, x_2^2, \dots, x_2^{J_2}\}.$$

Let $\mathbf{O}_{12} \equiv (X_1 \times X_2,\ \mathscr{P}(X_1 \times X_2),\ F \equiv F_1 \times^{\mathbf{O}_{12}} F_2)$ be a quasi-product observable with the marginal observables $\mathbf{O}_1$ and $\mathbf{O}_2$. (The existence of $\mathbf{O}_{12}$ is guaranteed by Lemma 2.7 since $C(\Omega)$ is commutative.) Put

$$\mathrm{Rep}[\mathbf{O}_{12}] = \begin{bmatrix} [F(\{(x_1^1, x_2^1)\})](\omega) & [F(\{(x_1^1, x_2^2)\})](\omega) & \dots & [F(\{(x_1^1, x_2^{J_2})\})](\omega) \\ [F(\{(x_1^2, x_2^1)\})](\omega) & [F(\{(x_1^2, x_2^2)\})](\omega) & \dots & [F(\{(x_1^2, x_2^{J_2})\})](\omega) \\ \vdots & \vdots & \ddots & \vdots \\ [F(\{(x_1^{J_1}, x_2^1)\})](\omega) & [F(\{(x_1^{J_1}, x_2^2)\})](\omega) & \dots & [F(\{(x_1^{J_1}, x_2^{J_2})\})](\omega) \end{bmatrix}.$$

Let $X = \{x^1, x^2, \dots, x^J\}$. Let $\mathbf{O} \equiv (X, \mathscr{P}(X), F)$ be an observable in a C^*-algebra $\mathscr{A}$. Put $X = \Xi_y \bigcup \Xi_n$ (where $\Xi_y \bigcap \Xi_n = \emptyset$). Define the map $g : X \to X_{(2)} \equiv \{y, n\}$ such that $g(x) = y$ (if $x \in \Xi_y$), $= n$ (if $x \in \Xi_n$). Here we can define the two-valued observable $(X_{(2)} \equiv \{y, n\}, \mathscr{P}(X_{(2)}), F_{(2)})$ in $\mathscr{A}$ as the image observable of g for $\mathbf{O}$. This two-valued observable is also called *yes–no observable* or *1–0 observable*. The following lemma lays the conditions that a quasi-product observable of yes–no observables should meet.

Lemma 3.1. *Consider yes–no observables* $\mathbf{O}_1 \equiv (X_1, \mathscr{P}(X_1), F_1)$ *and* $\mathbf{O}_2 \equiv (X_2, \mathscr{P}(X_2), F_2)$ *in a commutative* C^**-algebra* $C(\Omega)$ *such that*

$$X_1 = \{y_1, n_1\} \quad \textit{and} \quad X_2 = \{y_2, n_2\}.$$

Let $\mathbf{O}_{12} \equiv (X_1 \times X_2,\ \mathscr{P}(X_1 \times X_2),\ F \equiv F_1 \times^{\mathbf{O}_{12}} F_2)$ *be a quasi-product observable with the marginal observables* $\mathbf{O}_1$ *and* $\mathbf{O}_2$. *Put*

$$\begin{aligned} \mathrm{Rep}[\mathbf{O}_{12}] &= \begin{bmatrix} [F(\{(y_1, y_2)\})](\omega) & [F(\{(y_1, n_2)\})](\omega) \\ [F(\{(n_1, y_2)\})](\omega) & [F(\{(n_1, n_2)\})](\omega) \end{bmatrix} \\ &= \begin{bmatrix} \alpha(\omega) & [F_1(\{y_1\})](\omega) - \alpha(\omega) \\ [F_2(\{y_2\})](\omega) - \alpha(\omega) & 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega) \end{bmatrix}, \end{aligned} \tag{3.1}$$

where $\alpha \in C(\Omega)$. (*Note that* $[F(\{(y_1, y_2)\})](\omega) + [F(\{(y_1, n_2)\})](\omega) = [F_1(\{y_1\})](\omega)$ *and* $[F(\{(y_1, y_2)\})](\omega) + [F(\{(n_1, y_2)\})](\omega) = [F_2(\{y_2\})](\omega)$.) *Then,*

$$\max\{0, [F_1(\{y_1\})](\omega) + [F_2(\{y_2\})](\omega) - 1\} \leqslant \alpha(\omega) \leqslant \min\{[F_1(\{y_1\})](\omega),\ [F_2(\{y_2\})](\omega)\}$$
$$(\forall \omega \in \Omega). \tag{3.2}$$

Conversely, for any α $(\in C(\Omega))$ *that satisfies* (3.2), *the observable* $\mathbf{O}_{12}$ *defined by* (3.1) *is a quasi-product observable with the marginal observables* $\mathbf{O}_1$ *and* $\mathbf{O}_2$. *Also, note that*

$$[F(\{(y_1, n_2)\})](\omega) = 0 \ \Leftrightarrow\ \alpha(\omega) = [F_1(\{y_1\})](\omega) \ \Rightarrow\ [F_1(\{y_1\})](\omega) \leqslant [F_2(\{y_2\})](\omega). \tag{3.3}$$

Proof. Though this lemma is easy, we add a brief proof for completeness. Since $0 \leqslant [F(\{(x_1^1, x_2^2)\})](\omega) \leqslant 1$, $(\forall x^1, x^2 \in \{y, n\})$, we see, by (3.1), that

$$0 \leqslant \alpha(\omega) \leqslant 1, \quad 0 \leqslant [F_1(\{y_1\})](\omega) - \alpha(\omega) \leqslant 1, \quad 0 \leqslant [F_2(\{y_2\})](\omega) - \alpha(\omega) \leqslant 1,$$
$$0 \leqslant 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega) \leqslant 1, \tag{3.4}$$

which clearly implies (3.2). Conversely if α satisfies (3.2), then we easily see (3.4). Also, (3.3) is obvious. This completes the proof. □

Next we provide several examples, which will promote an understanding of our theory.

Example 3.2. Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ be a set of tomatos, which is regarded as a compact Hausdorff space with the discrete topology. Let $C(\Omega)$ be as in Example 2.2. Note that a tomato ω_n is represented by a fuzzy system $S_{\delta_{\omega_n}}$ (cf. Example 2.3). Consider yes–no observables $\mathbf{O}_{\mathrm{RD}} \equiv (X_{\mathrm{RD}}, \mathscr{P}(X_{\mathrm{RD}}), F_{\mathrm{RD}})$ and $\mathbf{O}_{\mathrm{SW}} \equiv (X_{\mathrm{SW}}, \mathscr{P}(X_{\mathrm{SW}}), F_{\mathrm{SW}})$ in $C(\Omega)$ such that

$$X_{\mathrm{RD}} = \{y_{\mathrm{RD}}, n_{\mathrm{RD}}\} \quad \text{and} \quad X_{\mathrm{SW}} = \{y_{\mathrm{SW}}, n_{\mathrm{SW}}\},$$

where we consider, for convenience sake (cf. Remark 3.3), that "y_{RD}" and "n_{RD}" respectively means "RED" and "NOT RED". Similarly, "y_{SW}" and "n_{SW}" respectively means "SWEET" and "NOT SWEET". We see, by Axiom 1, that

(∗) the probability that x_{RD} $(\in X_{\mathrm{RD}} \equiv \{y_{\mathrm{RD}}, n_{\mathrm{RD}}\})$, the measured value by the measurement $\mathbf{M}(\mathbf{O}_{\mathrm{RD}}, S_{\delta_{\omega_n}})$, belongs to Ξ_{RD} $(\subseteq X_{\mathrm{RD}} \equiv \{y_{\mathrm{RD}}, n_{\mathrm{RD}}\})$ is given by

$$\delta_{\omega_n}(F_{\mathrm{RD}}(\Xi_{\mathrm{RD}})) \quad (=[F_{\mathrm{RD}}(\Xi_{\mathrm{RD}})](\omega_n)).$$

That is, the probability that the tomato ω_n is observed as "RED" (resp. "NOT RED") is given by $[F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega_n)$ (resp. $[F_{\mathrm{RD}}(\{n_{\mathrm{RD}}\})](\omega_n)$). (Continued to Examples 3.4 and 4.14).

Remark 3.3. (i) The words (i.e., "RED" or "NOT RED") as labels are not essential in our theory. Recall Axiom 0, that is, "observable" is a fundamental object in measurements. Thus, it is proper that observables exist before words. In fact, many words were created in order to represent "observables". Therefore, though words are not essential in our theory, we do not deny the fact that "observable" (and thus, "membership function") is a handy mathematical tool for describing the ambiguity that can be found in the definition of a concept or the meaning of a word.

(ii) In order to promote a better understanding of our theory, we must provide a lot of examples, for instance, a quantum spin system ($\mathscr{A} = B(\mathbb{C}^2)$), a measurement of pencil's length by using fuzzy numbers ($\mathscr{A} = C(\mathbb{R} \cup \{\infty\})$), an "imaginary" measurement in statistical mechanics ($\mathscr{A} = C(\mathbb{R}^M \cup \{\infty\})$), etc.

(cf. [8]). In this paper we choose the simplicity of "tomato example" since we consider that it does not miss the essence of "fuzziness".

Example 3.4 (*Continued from Example* 3.2). Consider the quasi-product observable as follows:

$$\mathbf{O} = (X_{\mathrm{RD}} \times X_{\mathrm{SW}},\ \mathscr{P}(X_{\mathrm{RD}} \times X_{\mathrm{SW}}),\ F \equiv F_{\mathrm{RD}} \times^{\mathbf{O}} F_{\mathrm{SW}}),$$

that is,

$$\mathrm{Rep}[\mathbf{O}] = \begin{bmatrix} [F(\{(y_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega) & [F(\{(y_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega) \\ [F(\{(n_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega) & [F(\{(n_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(\omega) & [F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega) - \alpha(\omega) \\ [F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega) - \alpha(\omega) & 1 + \alpha(\omega) - [F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega) - [F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega) \end{bmatrix},$$

where $\alpha(\omega)$ satisfies (3.2). Hence by Axiom 1″, when we observe that the tomato ω_n is "RED", we can infer, by the fuzzy inference $\mathbf{M}(\mathbf{O}, S_{\delta_{\omega_n}})$, the probability that the tomato ω_n is "SWEET" is given by

$$\frac{[F(\{(y_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega_n)}{[F(\{(y_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega_n) + [F(\{(y_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega_n)}. \tag{3.5}$$

Here note that (3.5) implies

$$\text{"}[F(\{(y_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega_n) = 0\text{"} \quad \text{if and only if} \quad \text{"RED"} \Rightarrow \text{"SWEET"}. \tag{3.6}$$

which is also clearly equivalent to "NOT SWEET" ⇒ "NOT RED".

Being motivated by (3.6), we introduce the following definition of "implication" as a general form which is applicable to classical and quantum fuzzy systems.

Definition 3.5. Let $\mathbf{O}_1 \equiv (X_1, \mathscr{P}(X_1), F_1)$ and $\mathbf{O}_2 \equiv (X_2, \mathscr{P}(X_2), F_2)$ be observables (not necessarily two-valued observables) in a C^*-algebra $\mathscr{A}$. Let $\mathbf{O}_{12} = (X_1 \times X_2,\ \mathscr{P}(X_1 \times X_2),\ F_1 \times^{\mathbf{O}_{12}} F_2)$ be a quasi-product observable of $\mathbf{O}_1$ and $\mathbf{O}_2$. Let $\rho^p \in \mathfrak{S}^p(\mathscr{A}^*)$. Let $\Xi_1 \in \mathscr{P}(X_1)$ and $\Xi_2 \in \mathscr{P}(X_2)$. Then, the condition

$$\rho^p\Big((F_1 \times^{\mathbf{O}_{12}} F_2)(\Xi_1 \times (X_2 \setminus \Xi_2))\Big) = 0 \tag{3.7}$$

is denoted by

$$\mathbf{O}_1^{\Xi_1} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\rho^p})}{\Longrightarrow} \mathbf{O}_2^{\Xi_2}. \tag{3.8}$$

Remark 3.6. (i) (3.7) is of course also equal to

$$\mathbf{O}_1^{X_1 \setminus \Xi_1} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\rho^p})}{\Leftarrow} \mathbf{O}_2^{X_2 \setminus \Xi_2}$$

since $\mathbf{O}_{12} = \mathbf{O}_{\{1,2\}} = \mathbf{O}_{21}$ (i.e., $K = \{1,2\}$ is not regarded as an ordered set).

(ii) The data concerning both $\mathbf{O}_1$ and $\mathbf{O}_2$ for the system S_{ρ^p} are obtained by a simultaneous measurement $\mathbf{M}(\mathbf{O}_{12} \equiv \mathbf{O}_1 \times^{\mathbf{O}_{12}} \mathbf{O}_2, S_{\rho^p})$ (cf. Remark 2.11). Assume that we get a measured value (x_1, x_2) $(\in X_1 \times X_2)$ by the measurement $\mathbf{M}(\mathbf{O}_{12}, S_{\rho^p})$. And assume the condition (3.8). If we know that $x_1 \in \Xi_1$, then we can assure that $x_2 \in \Xi_2$. Therefore, Definition 3.5 is a direct consequence of Axiom 1′ (or, Axiom 1″).

Remark 3.7. (i) Since our theory is formulated in C^*-algebras (i.e., one of the fields of mathematics), it is proper that we always use "$\Rightarrow$" (as in (3.3) and elsewhere) as the rule in usual logic. That is, our theory is based on usual logic. The reader should not confuse this "$\Rightarrow$" with "$\underset{\mathbf{M}(\mathbf{O}_{12},S_{\rho^p})}{\Longrightarrow}$" in (3.8).

(ii) Note that physicists never use "quantum logic" but "usual logic" + "Born's axiom". In fact, most physicists are not familiar with quantum logic because it is simply one of the symbolic (and mathematical) aspects of Born's axiom (i.e., our axiom for $\mathscr{A} = \mathscr{C}(V)$ and crisp observables). Given our proposal for the foundation of measurements as in Section 2, our present situation is the same as that of physicists'. Hence we do not concern ourselves to make "fuzzy logic" as a mathematical symbolization of our axiom for $\mathscr{A} = C(\Omega)$. We are not concerned with mathematics but measurements (i.e., inferences). Note that mathematics has no reality in itself since mathematical theory always has a lot of interpretations. Therefore, our theory, as well as other excellent theories in physics, did not start from mathematics. Recall the correspondence "$\mathscr{M}(\mathscr{O}_Z, S_\Theta) \mapsto \mathbf{M}(\mathbf{O}, S_{\rho^p})$" in Axiom 0. Also recall the difference between the theory of differential equations and Newtonian mechanics.

4. Consistency and fuzzy syllogisms

In this section we study the consistent condition for observables (i.e., a generalization of Definition 2.6). We show several theorems of fuzzy syllogisms (i.e., theorems concerning "implication" in Definition 3.5) as a "fuzzy" aspect of our axiom. Following physicists' example, we do not intend to make "fuzzy logic" as a mathematical theory (cf. Remark 3.7).

Though we are not concerned with quantum theory in this paper, our investigations for classical systems become clearer in comparison with quantum theory (i.e., non-commutative fuzzy theory). Therefore, the following definitions (Definitions 4.1 and 4.2) are common in both classical and quantum fuzzy theory.

Definition 4.1. Let $\mathscr{A}$ be a C^*-algebra. For each $k \in K \equiv \{1,2,\ldots,|K|-1,|K|\}$, consider a label set X_k. Consider $\mathscr{D}$ $(\subseteq \mathscr{P}(K))$ such that $\bigcup_{D\in\mathscr{D}} D = K$. Then, $\mathscr{G} \equiv [\mathbf{O}_D \equiv (\times_{k\in D} X_k,\ \mathscr{P}(\times_{k\in D} X_k),\ F_D) : D\in\mathscr{D}]$ is called a *covering family of observables in* $\mathscr{A}$, if it satisfies the following condition :

$$\mathbf{O}_{D_1}|_{D_1\cap D_2} = \mathbf{O}_{D_2}|_{D_1\cap D_2} \quad (\forall D_1, \forall D_2 \in \mathscr{D} \text{ such that } D_1\cap D_2 \neq \emptyset).$$

Note that, if $\mathscr{G}$ is a covering family, then $\mathbf{O}_{D_1}|_{\{k\}} = \mathbf{O}_{D_2}|_{\{k\}}$ for any $k\in K$ and any $D_1, D_2 \in \mathscr{D}$ such that $k \in D_1 \bigcap D_2$. Thus, a covering family of observables $\mathscr{G}$ determines a unique $\{k\}$-marginal observable $\mathbf{O}_k \equiv (X_k, \mathscr{P}(X_k), F_k)$ for each $k\in K$.

The following definition is a generalization of Definition 2.6 (i.e., the case that $\mathscr{D} = \{\{1\},\{2\},\ldots,\{|K|\}\}$).

Definition 4.2. Let $\mathscr{A}$ be a C^*-algebra. A covering family of observable $\mathscr{G} \equiv [\ \mathbf{O}_D \equiv (\times_{k\in D} X_k,\ \mathscr{P}(\times_{k\in D} X_k),\ F_D) : D\in\mathscr{D}\ (\subseteq\mathscr{P}(K))]$ in $\mathscr{A}$ is called *consistent*, if there exists an observable $\mathbf{O}_K \equiv (\times_{k\in K} X_k,\ \mathscr{P}(\times_{k\in K} X_k),\ F)$ in $\mathscr{A}$ such that

$$\mathbf{O}_K|_D = \mathbf{O}_D \quad (\forall D\in\mathscr{D}). \tag{4.1}$$

Also, the relation (4.1) is denoted by

$$[\mathbf{O}_D : D\in\mathscr{D}] \sqsubset \mathbf{O}_K. \tag{4.2}$$

Remark 4.3. Under the condition (4.2), the data concerning $\mathscr{G} \equiv [\mathbf{O}_D : D \in \mathscr{D}]$ for the system S_{ρ^p} is obtained by the simultaneous measurement $\mathbf{M}(\mathbf{O}_K, S_{\rho^p})$. So a covering family $\mathscr{G}$ has no reality, if it is not consistent. Recall the arguments in Remark 2.11, which correspond to the above definition for the case $\mathscr{D} = \{\{1\},\{2\}\}$.

Lemma 4.4. *Let $\mathscr{A}$ be a C^*-algebra. Let $\mathscr{G}_1 \equiv [\mathbf{O}^1_{D_1} : D_1 \in \mathscr{D}_1(\subseteq \mathscr{P}(K))]$ be a covering family of observables in $\mathscr{A}$ and $\mathscr{G}_2 \equiv [\mathbf{O}^2_{D_2} : D_2 \in \mathscr{D}_2(\subseteq \mathscr{P}(K))]$ be a consistent covering family of observables in $\mathscr{A}$. Assume that for any $D_1 \in \mathscr{D}_1$ there exists a D_2 $(\in \mathscr{D}_2)$ such that*

$$D_1 \subseteq D_2 \quad \text{and} \quad \mathbf{O}^1_{D_1} = \mathbf{O}^2_{D_2}|_{D_1}. \tag{4.3}$$

Then, $\mathscr{G}_1$ is consistent.

Proof. Since a covering family $\mathscr{G}_2$ is consistent, there exists an observable $\mathbf{O}_K \equiv (\times_{k\in K} X_k, \mathscr{P}(\times_{k\in K} X_k), F_K)$ in $\mathscr{A}$ such that $\mathbf{O}^2_{D_2} = \mathbf{O}_K|_{D_2}$ $(\forall D_2 \in \mathscr{D}_2)$. Let D_1 be any element in $\mathscr{D}_1$. Then, by choosing D_2 $(\in \mathscr{D}_2)$ satisfying (4.3), we see that $\mathbf{O}^1_{D_1} = \mathbf{O}^2_{D_2}|_{D_1} = (\mathbf{O}_K|_{D_2})|_{D_1} = \mathbf{O}_K|_{D_1}$. This completes the proof. □

Lemma 4.5. *Let $\mathscr{A}$ be a commutative C^*-algebra (i.e., $\mathscr{A} = C(\Omega)$). Let D_{12} and D_{23} be subsets of K. Put $D_{123} \equiv D_{12} \bigcup D_{23} \equiv (D_{12} \setminus D_{23}) \bigcap (D_{12} \cap D_{23}) \bigcap (D_{23} \setminus D_{12}) \equiv D_1 \bigcup D_2 \bigcup D_3$. Consider the following observables in $C(\Omega)$:*

$$\mathbf{O}_{D_{12}} \equiv \left(\mathop{\times}_{k\in D_{12}} X_k,\ \mathscr{P}\left(\mathop{\times}_{k\in D_{12}} X_k \right),\ F_{D_{12}} \right) \quad \text{and} \quad \mathbf{O}_{D_{23}} \equiv \left(\mathop{\times}_{k\in D_{23}} X_k,\ \mathscr{P}\left(\mathop{\times}_{k\in D_{23}} X_k \right),\ F_{D_{23}} \right)$$

such that $\mathbf{O}_{D_{12}}|_{D_2} = \mathbf{O}_{D_{23}}|_{D_2}$. Then, there exists an observable $\mathbf{O}_{D_{123}} \equiv (\times_{k\in D_{123}} X_k, \mathscr{P}(\times_{k\in D_{123}} X_k), F_{D_{123}})$ such that $\mathbf{O}_{D_{123}}|_{D_{12}} = \mathbf{O}_{D_{12}}$ and $\mathbf{O}_{D_{123}}|_{D_{23}} = \mathbf{O}_{D_{23}}$.

Proof. Assume that $D_{12} \bigcap D_{23} \neq \emptyset$. (If $D_{12} \bigcap D_{23} = \emptyset$, this lemma is an immediate consequence of Lemma 2.7.) Put $Y_m = \times_{k\in D_m} X_k = \{y_m^1, y_m^2, \ldots, y_m^{j_m}, \ldots, y_m^{M_m}\}$, $m = 1,2,3$. (So, $M_m = \prod_{k\in D_m} |X_k|$.) Thus, we can put, by $Y_1 \times Y_2 = \times_{k\in D_{12}} X_k$ and $Y_2 \times Y_3 = \times_{k\in D_{23}} X_k$, that

$$\mathbf{O}_{D_{12}} = (Y_1 \times Y_2, \mathscr{P}(Y_1 \times Y_2), F_{12} \equiv F_{D_{12}})$$

and

$$\mathbf{O}_{D_{23}} = (Y_2 \times Y_3, \mathscr{P}(Y_2 \times Y_3), F_{23} \equiv F_{D_{23}}).$$

Define the observable $\mathbf{O}_{D_{123}} \equiv (\times_{m=1}^3 Y_m, \mathscr{P}(\times_{m=1}^3 Y_m), F_{123})$ in $C(\Omega)$ such that

$$[F_{123}(\{(y_1^{j_1}, y_2^{j_2}, y_3^{j_3})\})](\omega) = \begin{cases} \dfrac{[F_{12}(\{(y_1^{j_1}, y_2^{j_2})\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)} & \text{if } [F_2(\{y_2^{j_2}\})](\omega) \neq 0, \\ 0 & \text{if } [F_2(\{y_2^{j_2}\})](\omega) = 0, \end{cases}$$

for $1 \leqslant \forall j_1 \leqslant M_1$, $1 \leqslant \forall j_2 \leqslant M_2$, $1 \leqslant \forall j_3 \leqslant M_3$. Therefore, it is clear that this lemma holds. For example, $\mathbf{O}_{D_{123}}|_{D_{23}} = \mathbf{O}_{D_{23}}$ is easily seen as follows:

$$\begin{aligned} [F_{123}(Y_1 \times \{(y_2^{j_2}, y_3^{j_3})\})](\omega) &= \sum_{y_1^{j_1}\in Y_1} [F_{123}(\{(y_1^{j_1}, y_2^{j_2}, y_3^{j_3})\})](\omega) \\ &= \sum_{y_1^{j_1}\in Y_1} \frac{[F_{12}(\{(y_1^{j_1}, y_2^{j_2})\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)} \end{aligned}$$

$$= \frac{[F_{12}(Y_1 \times \{y_2^{j_2}\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)}$$

$$= \frac{[F_2(\{y_2^{j_2}\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)}$$

$$= [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega) \quad (\forall \omega \in \Omega,\ 1 \leqslant \forall j_2 \leqslant M_2,\ 1 \leqslant \forall j_3 \leqslant M_3).$$

This completes the proof.

The following theorem is a kind of generalizations of Lemma 2.7 (which essentially corresponds to the result for $\mathscr{D} = \{\{1\}, \{2\}, \dots, \{|K|\}\}$ in the following theorem). Here note that a covering family $[\mathbf{O}_D : D \in \mathscr{D}]$ is equivalent to $[\mathbf{O}_{D'} : D' \in \{D' : D' \subseteq D \text{ for some } D \in \mathscr{D}\}]$ where $\mathbf{O}_{D'} = \mathbf{O}_D|_{D'}$ for any D' such that $D' \subseteq D$.

Theorem 4.6. *Let $\mathscr{D} = \{\{1,2\}, \{2,3\}, \dots, \{|K|-1, |K|\}\}$ $(\subseteq \mathscr{P}(K))$. Let $\mathscr{G} = [\mathbf{O}_D = (\times_{k \in D} X_k, \mathscr{P}(\times_{k \in D} X_k), F_D) : D \in \mathscr{D}]$ be a covering family of observables in a commutative C^*-algebra $C(\Omega)$. (Here we can put $\mathscr{G} = [\mathbf{O}_{k,k+1} \equiv (X_k \times X_{k+1}, \mathscr{P}(X_k \times X_{k+1}), F_{k,k+1} \equiv F_k \times^{\mathbf{O}_{k,k+1}} F_{k+1}) : k = 1,2,\dots,|K|-1]$. Then, $\mathscr{G} = [\mathbf{O}_{k,k+1} : k = 1,2,\dots,|K|-1]$ is consistent.*

Proof. Put $D_{12} = \{1,2\}$ and $D_{23} = \{2,3\}$. By Lemma 4.5, we get $\mathbf{O}_{123}$ $(= \mathbf{O}_{D_{123}})$ such that $\mathscr{G}_3 = [\mathbf{O}_{123}, \mathbf{O}_{34}, \mathbf{O}_{45}, \dots, \mathbf{O}_{|K|-1,|K|}]$ is a covering family in $C(\Omega)$, where $\mathbf{O}_{12} = \mathbf{O}_{123}|_{\{1,2\}}$ and $\mathbf{O}_{23} = \mathbf{O}_{123}|_{\{2,3\}}$. Iteratively, we get $\mathscr{G}_{|K|-1} = [\mathbf{O}_{123\dots|K|-1}, \mathbf{O}_{|K|-1,|K|}]$ and $\mathscr{G}_{|K|} = [\mathbf{O}_{123\dots|K|-1,|K|}] \equiv [\mathbf{O}_K]$, which is clearly consistent. So, by Lemma 4.4, we see that $\mathscr{G}_{|K|-1} \sqsubset \mathbf{O}_K$. Therefore, we iteratively get $\mathscr{G} \sqsubset \mathbf{O}_K$. This completes the proof. □

Remark 4.7. This theorem is due to the commutativity of a C^*-algebra $C(\Omega)$. In general (particularly in quantum systems, i.e., $\mathscr{A} = \mathscr{C}(V)$), there exists no $\mathbf{O}_{123}$ such that $[\mathbf{O}_{12}, \mathbf{O}_{23}] \sqsubset \mathbf{O}_{123}$ (i.e., $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ is not consistent in general). Thus, we have no simultaneous measurement $\mathbf{M}(\mathbf{O}_{123}, S_{\rho^p})$. Therefore, in general, we cannot get information (i.e., data) concerning the covering family $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ for the quantum system S_{ρ^p}. That is, in general, the covering family $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ has no reality in quantum mechanics.

The following notation is the preparation for Theorems 4.12 and 4.16.

Notation 4.8. Let $\mathscr{G} = [\mathbf{O}_{12}, \mathbf{O}_{23}, \dots \mathbf{O}_{|K|-1,|K|}] \equiv [(X_k \times X_{k+1}, \mathscr{P}(X_k \times X_{k+1}), F_{k,k+1} \equiv F_k \times^{\mathbf{O}_{k,k+1}} F_{k+1}) : k = 1,2,\dots,|K|-1]$ be a covering family of observables in a commutative C^*-algebra $C(\Omega)$. (So, $\mathscr{G}$ is consistent as in Theorem 4.6.) Suppose that $X_k = \{y_k, n_k\}$ for each $k \in K$. As in Definition 4.1, put

$$\mathrm{Rep}[\mathbf{O}_k] = \mathrm{Rep}[(X_k, \mathscr{P}(X_k), F_k)] = \Big[[F_k(\{y_k\})](\omega), [F_k(\{n_k\})](\omega)\Big] \equiv \Big[p_k^1(\omega), p_k^0(\omega)\Big]$$

for all $k = 1,2,3,\dots,|K|$. Furthermore, put

$$\begin{aligned}
\mathrm{Rep}[\mathbf{O}_{k,k+1}] &= \mathrm{Rep}[(X_k \times X_{k+1},\ \mathscr{P}(X_k \times X_{k+1}),\ F_{k,k+1})] \\
&= \begin{bmatrix} [F_{k,k+1}(\{y_k\}\times\{y_{k+1}\})](\omega) & [F_{k,k+1}(\{y_k\}\times\{n_{k+1}\})](\omega) \\ [F_{k,k+1}(\{n_k\}\times\{y_{k+1}\})](\omega) & [F_{k,k+1}(\{n_k\}\times\{n_{k+1}\})](\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p_{k,k+1}^{11}(\omega) & p_{k,k+1}^{10}(\omega) \\ p_{k,k+1}^{01}(\omega) & p_{k,k+1}^{00}(\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p_{k,k+1}^{11}(\omega) & p_k^1(\omega) - p_{k,k+1}^{11}(\omega) \\ p_{k+1}^1(\omega) - p_{k,k+1}^{11}(\omega) & 1 + p_{k,k+1}^{11}(\omega) - p_k^1(\omega) - p_{k+1}^1(\omega) \end{bmatrix}
\end{aligned} \tag{4.4}$$

for all $k = 1,2,\dots,|K|-1$, where $p^{11}_{k,k+1}(\omega)$ satisfies (3.2). Let $\mathbf{O}_K \equiv (\times_{k\in K} X_k,\ \mathscr{P}(\times_{k\in K} X_k),\ F_K)$ be any observable in $C(\Omega)$ such that

$$[\mathbf{O}_{12}, \mathbf{O}_{23}, \dots \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K. \tag{4.5}$$

(The existence of $\mathbf{O}_K$ is guaranteed by Theorem 4.6.) Put

$$\left[p^{j_1,j_2,\dots,j_{|K|}}_{1,2,\dots,|K|}(\omega) : j_1,j_2,\dots,j_{|K|} = 1,0\right] \equiv \left[\left[F_K\left(\mathop{\times}_{k=1}^{|K|}\{x_k^{j_k}\}\right)\right](\omega) : j_1,j_2,\dots,j_{|K|} = 1,0\right], \tag{4.6}$$

where $x_k^{j_k} = y_k$ (if $j_k = 1$) and $x_k^{j_k} = n_k$ (if $j_k = 0$). Define $\mathbf{O}_{1,|K|} \equiv (X_1 \times X_{|K|},\ \mathscr{P}(X_1 \times X_{|K|}),\ F_{1,|K|})$ such that $\mathbf{O}_{1,|K|} = \mathbf{O}_K|_{\{1,|K|\}}$. Put

$$\begin{aligned}
\mathrm{Rep}[\mathbf{O}_{1,|K|}] &= \mathrm{Rep}[(X_1 \times X_{|K|},\ \mathscr{P}(X_1 \times X_{|K|}),\ F_{1,|K|})] \\
&= \begin{bmatrix} [F_{1,|K|}(\{y_1\}\times\{y_{|K|}\})](\omega) & [F_{1,|K|}(\{y_1\}\times\{n_{|K|}\})](\omega) \\ [F_{1,|K|}(\{n_1\}\times\{y_{|K|}\})](\omega) & [F_{1,|K|}(\{n_1\}\times\{n_{|K|}\})](\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p^{11}_{1,|K|}(\omega) & p^{10}_{1,|K|}(\omega) \\ p^{01}_{1,|K|}(\omega) & p^{00}_{1,|K|}(\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p^{11}_{1,|K|}(\omega) & p^{1}_{1}(\omega) - p^{11}_{1,|K|}(\omega) \\ p^{1}_{|K|}(\omega) - p^{11}_{1,|K|}(\omega) & 1 + p^{11}_{1,|K|}(\omega) - p^{1}_{1}(\omega) - p^{1}_{|K|}(\omega) \end{bmatrix}
\end{aligned} \tag{4.7}$$

(Continued to Lemmas 4.9 and 4.10 and Theorem 4.12 for $K = \{1,2,3\}$, and to Theorem 4.16 for general case).

Lemma 4.9. *Using Notation* 4.8 *for* $K = \{1,2,3\}$, *we see* (*putting* $p^{j_1j_2j_3}_{123} = p^{j_1j_2j_3}_{123}(\omega)$ *in* (4.6), $p^{111}_{123} = A$ *and* $p^{101}_{123} = B$),

$$\begin{aligned}
&p^{111}_{123} = A(\omega), && p^{011}_{123} = p^{11}_{23} - A(\omega), \\
&p^{110}_{123} = p^{11}_{12} - A(\omega), && p^{010}_{123} = p^{1}_{2} - p^{11}_{12} - p^{11}_{23} + A(\omega), \\
&p^{101}_{123} = B(\omega), && p^{001}_{123} = p^{1}_{3} - p^{11}_{23} - B(\omega), \\
&p^{100}_{123} = p^{1}_{1} - p^{11}_{12} - B(\omega), && p^{000}_{123} = 1 - p^{1}_{1} - p^{1}_{2} - p^{1}_{3} + p^{11}_{12} + p^{11}_{23} + B(\omega),
\end{aligned} \tag{4.8}$$

where

$$\max\{0, -p^1_2(\omega) + p^{11}_{12}(\omega) + p^{11}_{23}(\omega)\} \leqslant A(\omega) \leqslant \min\{p^{11}_{12}(\omega), p^{11}_{23}(\omega)\} \tag{4.9}$$

and

$$\begin{aligned}
&\max\{0, p^1_1(\omega) + p^1_2(\omega) + p^1_3(\omega) - p^{11}_{12}(\omega) - p^{11}_{23}(\omega) - 1\} \\
&\quad \leqslant B(\omega) \leqslant \min\{p^1_1(\omega) - p^{11}_{12}(\omega), p^1_3(\omega) - p^{11}_{23}(\omega)\}.
\end{aligned} \tag{4.10}$$

Proof. From (4.6), (4.5) and (4.4) for $K = \{1,2,3\}$, we see

$$\begin{aligned}
&p^{111}_{123} + p^{110}_{123} = p^{11}_{12}, && p^{101}_{123} + p^{100}_{123} = p^{10}_{12} = p^1_1 - p^{11}_{12}, \\
&p^{011}_{123} + p^{010}_{123} = p^{01}_{12} = p^1_2 - p^{11}_{12}, && p^{001}_{123} + p^{000}_{123} = p^{00}_{12} = 1 + p^{11}_{12} - p^1_1 - p^1_2,
\end{aligned}$$

$$p_{123}^{111} + p_{123}^{011} = p_{23}^{11}, \qquad p_{123}^{110} + p_{123}^{010} = p_{23}^{10} = p_2^1 - p_{23}^{11},$$
$$p_{123}^{101} + p_{123}^{001} = p_{23}^{01} = p_3^1 - p_{23}^{11}, \qquad p_{123}^{100} + p_{123}^{000} = p_{23}^{00} = 1 - p_{23}^{11} - p_2^1 - p_3^1.$$

After a small computation, we get (4.8). Since $0 \leqslant p_{123}^{j_1j_2j_3}(\omega) \leqslant 1$, we see, from (4.8), that

$$0 \leqslant A \leqslant 1, \qquad p_{23}^{11} - 1 \leqslant A \leqslant p_{23}^{11}, \qquad p_{12}^{11} - 1 \leqslant A \leqslant p_{12}^{11},$$
$$-p_2^1 + p_{12}^{11} + p_{23}^{11} \leqslant A \leqslant 1 - p_2^1 + p_{12}^{11} + p_{23}^{11},$$
$$0 \leqslant B \leqslant 1, \qquad p_3^1 - p_{23}^{11} - 1 \leqslant B \leqslant p_3^1 - p_{23}^{11}, \qquad p_1^1 - p_{12}^{11} - 1 \leqslant B \leqslant p_1^1 - p_{12}^{11},$$
$$p_1^1 + p_2^1 + p_3^1 - p_{12}^{11} - p_{23}^{11} - 1 \leqslant B \leqslant p_1^1 + p_2^1 + p_3^1 - p_{12}^{11} - p_{23}^{11}.$$

This implies (4.9) and (4.10). This completes the proof. □

Lemma 4.10. *Using Notation* 4.8 *for* $K = \{1,2,3\}$, *we see*

$$\begin{aligned} &\max\{0, -p_2^1(\omega) + p_{12}^{11}(\omega) + p_{23}^{11}(\omega)\} \\ &\quad + \max\{0, p_1^1(\omega) + p_2^1(\omega) + p_3^1(\omega) - p_{12}^{11}(\omega) - p_{23}^{11}(\omega) - 1\} \\ &\qquad \leqslant p_{13}^{11}(\omega) \qquad (4.11) \\ &\qquad \leqslant \min\{p_{12}^{11}(\omega), p_{23}^{11}(\omega)\} + \min\{p_1^1(\omega) - p_{12}^{11}(\omega), p_3^1(\omega) - p_{23}^{11}(\omega)\}. \qquad (4.12) \end{aligned}$$

Proof. Since $p_{13}^{11}(\omega) = p_{123}^{111}(\omega) + p_{123}^{101}(\omega) = A(\omega) + B(\omega)$ in Lemma 4.9, by (4.9) and (4.10) we can easily get (4.11) and (4.12). This completes the proof. □

Remark 4.11. Let us compare the result in Lemma 4.10 with (3.2) (i.e., without consistency condition). Note that (3.2) implies

$$C_1 \equiv \max\{0, p_1^1(\omega) + p_3^1(\omega) - 1\} \leqslant p_{13}^{11}(\omega) \leqslant \min\{p_1^1(\omega), p_3^1(\omega)\} \equiv C_2.$$

Here we can easily see that $C_1 \leqslant$ (4.11) and (4.12) $\leqslant C_2$ from the following trivial inequalities:

$$\max\{0, \alpha_1 + \alpha_2\} \leqslant \max\{0, \max\{0, \alpha_1\} + \max\{0, \alpha_2\}\} = \max\{0, \alpha_1\} + \max\{0, \alpha_2\}$$

and

$$\min\{\alpha_1, \alpha_2\} + \min\{\alpha_3, \alpha_4\} = \min\{\alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\} \leqslant \min\{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4\}.$$

Therefore, we see in Lemma 4.10 that the value $p_{13}^{11}(\omega)$ is restricted under the consistent condition of $[\mathbf{O}_{12}, \mathbf{O}_{23}]$.

Now we show several theorems of fuzzy syllogisms (i.e., theorems concerning "implication" in Definition 3.5) as the consequences of our arguments.

Theorem 4.12 (Fuzzy syllogism). *Using Notation* 4.8 *for* $K = \{1,2,3\}$, *we have that*, $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ *is a covering family of observables in a commutative* C^**-algebra* $C(\Omega)$. *Let* $\delta_{\omega_0} \in \mathcal{M}_{+1}^p(\Omega)$ *for any fixed* $\omega_0 \in \Omega$. *Let* $\mathbf{O}_{123}$ $(=\mathbf{O}_K)$ *be any observable such that* $[\mathbf{O}_{12}, \mathbf{O}_{23}] \sqsubset \mathbf{O}_{123}$ *and let* $\mathbf{O}_{13} = \mathbf{O}_{123}|_{\{1,3\}}$. (*The existence of* $\mathbf{O}_{123}$ *is guaranteed by Theorem* 4.6.) *Then we have the following*:

(1) *Assume that*

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}(\mathbf{O}_{23}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \qquad (4.13)$$

Then, we see that

$$\begin{bmatrix} p_{13}^{11}(\omega_0) & p_{13}^{10}(\omega_0) \\ p_{13}^{01}(\omega_0) & p_{13}^{00}(\omega_0) \end{bmatrix} = \begin{bmatrix} p_1^1(\omega_0) & 0 \\ p_3^1(\omega_0) - p_1^1(\omega_0) & 1 - p_3^1(\omega_0) \end{bmatrix}; \tag{4.14}$$

hence, we see that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{13}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{4.15}$$

(2) *Assume that*

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\delta_{\omega_0}})}{\Longleftarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}(\mathbf{O}_{23}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{4.16}$$

Then, we see that

$$\begin{bmatrix} p_{13}^{11}(\omega_0) & p_{13}^{10}(\omega_0) \\ p_{13}^{01}(\omega_0) & p_{13}^{00}(\omega_0) \end{bmatrix} = \begin{bmatrix} \alpha(\omega_0) & p_1^1(\omega_0) - \alpha(\omega_0) \\ p_3^1(\omega_0) - \alpha(\omega_0) & 1 + \alpha(\omega_0) - p_1^1(\omega_0) - p_3^1(\omega_0) \end{bmatrix},$$

where

$$\max\{p_2^1(\omega_0), p_1^1(\omega_0) + p_3^1(\omega_0) - 1\} \leqslant \alpha(\omega_0) \leqslant \min\{p_1^1(\omega_0), p_3^1(\omega_0)\}. \tag{4.17}$$

Also (4.16) *is equivalent to*

$$\mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}(\mathbf{O}_{123}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_{13}^{\{(y_1, y_3)\}}. \tag{4.18}$$

(3) *Assume that*

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}(\mathbf{O}_{23}, S_{\delta_{\omega_0}})}{\Leftarrow} \mathbf{O}_3^{\{y_3\}}. \tag{4.19}$$

Then, we see that

$$\begin{bmatrix} p_{13}^{11}(\omega_0) & p_{13}^{10}(\omega_0) \\ p_{13}^{01}(\omega_0) & p_{13}^{00}(\omega_0) \end{bmatrix} = \begin{bmatrix} \alpha(\omega_0) & p_1^1(\omega_0) - \alpha(\omega_0) \\ p_3^1(\omega_0) - \alpha(\omega_0) & 1 + \alpha(\omega_0) - p_1^1(\omega_0) - p_3^1(\omega_0) \end{bmatrix},$$

where

$$\max\{0, p_1^1(\omega_0) + p_3^1(\omega_0) - p_2^1(\omega_0)\} \leqslant \alpha(\omega_0) \leqslant \min\{p_1^1(\omega_0), p_3^1(\omega_0)\}. \tag{4.20}$$

Also (4.19) *is equivalent to*

$$\mathbf{O}_{13}^{\{(y_1, y_3), (y_1, n_3), (n_1, y_3)\}} \underset{\mathbf{M}(\mathbf{O}_{123}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}. \tag{4.21}$$

Proof. (1) By (4.13) and (3.3), we see that $p_{12}^{10} = p_{23}^{10} = 0$, so, $p_{12}^{11} = p_1^1 \leqslant p_2^1 = p_{23}^{11} \leqslant p_3^1$. Therefore, we see that (4.11) $= p_{12}^{11} + \max\{0, p_3^1 - 1\} = p_1^1$, and (4.12) $= p_1^1 + 0 = p_1^1$. This implies that $p_{13}^{11} = p_1^1$, i.e., (4.14). Also, (4.15) follows from $P_{13}^{10} = 0$.

(2) By (4.16) and (3.3), we see that $p_{12}^{01} = p_{23}^{10} = 0$, so, $p_{12}^{11} = p_2^1 \leqslant p_1^1$ and $p_{23}^{11} = p_2^1 \leqslant p_3^1$. Therefore, we see that (4.11) $= p_{23}^{11} + \max\{0, p_1^1 - p_2^1 + p_3^1 - 1\} = \max\{p_2^1, p_1^1 + p_3^1 - 1\}$, and (4.12) $= \min\{p_2^1, p_2^1\} + \min\{p_1^1 - p_2^1, p_3^1 - p_2^1\} = \min\{p_1^1, p_3^1\}$. This implies (4.17). Also, we see that (4.16) $\Leftrightarrow p_{12}^{01} = p_{23}^{10} = 0 \Leftrightarrow p_{123}^{010} = p_{123}^{011} = p_{123}^{110} = 0 \Leftrightarrow$ (4.18).

(3) By (4.19) and (3.3), we see that $p_{12}^{10} = p_{23}^{01} = 0$, so, $p_{12}^{11} = p_1^1 \leqslant p_2^1$ and $p_{23}^{11} = p_3^1 \leqslant p_2^1$. Therefore, we see that (4.11) $= \max\{0, p_1^1 - p_2^1 + p_3^1\} + \max\{0, p_2^1 - 1\} = \max\{0, p_1^1 - p_2^1 + p_3^1\}$. And (4.12) $= \min\{p_1^1, p_3^1\}$. This implies (4.20). Also, (4.19) $\Leftrightarrow p_{12}^{10} = p_{23}^{01} = 0 \Leftrightarrow p_{123}^{101} = p_{123}^{100} = p_{123}^{001} = 0 \Leftrightarrow$ (4.21). This completes the proof. □

Remark 4.13. The reader must not confuse the result (for example, (4.13)⇒(4.15)) in Theorem 4.12 with a rule in mathematics (or logic). Theorem 4.12 is a consequence of our axiom. Note that Theorem 4.12 is due to Theorem 4.6, i.e., the commutativity of C^*-algebra $C(\Omega)$ (cf. Remark 4.7). That means the results in Theorem 4.12 cannot be expected in quantum systems. For example, "(4.13)⇒(4.15)" is meaningless in quantum systems. In comparison with quantum theory, Theorem 4.12 becomes clearer.

Example 4.14. (*Continued from Example* 3.2). Let Ω, $C(\Omega)$, $\mathbf{O}_1 \equiv \mathbf{O}_{\mathrm{SW}} \equiv (X_{\mathrm{SW}}, \mathscr{P}(X_{\mathrm{SW}}), F_{\mathrm{SW}})$ and $\mathbf{O}_3 \equiv \mathbf{O}_{\mathrm{RD}} \equiv (X_{\mathrm{RD}}, \mathscr{P}(X_{\mathrm{RD}}), F_{\mathrm{RD}})$ be as in Example 3.2. Let $\mathbf{O}_2 \equiv \mathbf{O}_{\mathrm{RP}} \equiv (X_{\mathrm{RP}}, \mathscr{P}(X_{\mathrm{RP}}), F_{\mathrm{RP}})$ be an observable in $C(\Omega)$ such that

$$X_{\mathrm{RP}} = \{y_{\mathrm{RP}}, n_{\mathrm{RP}}\},$$

where "y_{RP}" and "n_{RP}" respectively means "RIPE" and "NOT RIPE". Put

$$\mathrm{Rep}[\mathbf{O}_1] = \Big[[F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega), [F_{\mathrm{SW}}(\{n_{\mathrm{SW}}\})](\omega)\Big],$$

$$\mathrm{Rep}[\mathbf{O}_2] = \Big[[F_{\mathrm{RP}}(\{y_{\mathrm{RP}}\})](\omega), [F_{\mathrm{RP}}(\{n_{\mathrm{RP}}\})](\omega)\Big],$$

$$\mathrm{Rep}[\mathbf{O}_3] = \Big[[F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega), [F_{\mathrm{RD}}(\{n_{\mathrm{RD}}\})](\omega)\Big].$$

Consider the following quasi-product observables:

$$\mathbf{O}_{12} = (X_{\mathrm{SW}} \times X_{\mathrm{RP}}, \mathscr{P}(X_{\mathrm{SW}} \times X_{\mathrm{RP}}), F_{12} \equiv F_{\mathrm{SW}} \times^{\mathbf{O}_{12}} F_{\mathrm{RP}})$$

and

$$\mathbf{O}_{23} = (X_{\mathrm{RP}} \times X_{\mathrm{RD}}, \mathscr{P}(X_{\mathrm{RP}} \times X_{\mathrm{RD}}), F_{23} \equiv F_{\mathrm{RP}} \times^{\mathbf{O}_{23}} F_{\mathrm{RD}}).$$

Let $\delta_{\omega_n} \in \mathscr{M}_{+1}^{p}(\Omega)$ for any fixed $\omega_n \in \Omega$. Assume that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}(\mathbf{O}_{23}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{4.22}$$

Then, we see, by Theorem 4.12(1), that

$$\mathrm{Rep}[\mathbf{O}_{13}] = \begin{bmatrix} [F_{13}(\{y_{\mathrm{SW}}\} \times \{y_{\mathrm{RD}}\})](\omega_n) & [F_{13}(\{y_{\mathrm{SW}}\} \times \{n_{\mathrm{RD}}\})](\omega_n) \\ [F_{13}(\{n_{\mathrm{SW}}\} \times \{y_{\mathrm{RD}}\})](\omega_n) & [F_{13}(\{n_{\mathrm{SW}}\} \times \{n_{\mathrm{RD}}\})](\omega_n) \end{bmatrix}$$
$$= \begin{bmatrix} [F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega_n) & 0 \\ [F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega_n) - [F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega_n) & 1 - [F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega_n) \end{bmatrix}. \tag{4.23}$$

So, when we observe that the tomato ω_n is "RED", we can infer, by the fuzzy inference $\mathbf{M}(\mathbf{O}_{13}, S_{\delta_{\omega_n}})$ (equivalently, $\mathbf{M}(\mathbf{O}_{31}, S_{\delta_{\omega_n}})$), the probability that the tomato ω_n is "SWEET" is given by

$$\frac{[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)}{[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)+[F_{13}(\{n_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)}=\frac{[F_{\rm SW}(\{y_{\rm SW}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}. \tag{4.24}$$

Also, when we observe that the tomato ω_n is "SWEET", we can infer, by the fuzzy inference $\mathbf{M}(\mathbf{O}_{13}, S_{\delta_{\omega_n}})$, the probability that the tomato ω_n is "RED" is given by

$$\frac{[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)}{[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)+[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{n_{\rm RD}\})](\omega_n)}=\frac{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}=1. \tag{4.25}$$

Note that (4.22) respectively implies (and is implied by)

$$\text{"SWEET"} \Rightarrow \text{"RIPE"} \quad \text{and} \quad \text{"RIPE"} \Rightarrow \text{"RED"}. \tag{4.26}$$

(Recall (3.6).) So, it is "reasonable" to reach the conclusion:

$$\text{"SWEET"} \Rightarrow \text{"RED"}, \tag{4.27}$$

which is implied by (4.25). (Here we are afraid that the most important fact may be overlooked. For completeness, note that the conclusion "(4.26) $\Rightarrow$ (4.27)" is a consequence of Theorem 4.12 (and therefore, of our axiom). Also see Remark 4.17.) However, (4.24) is due to the peculiarity of fuzzy inferences. That is, in spite of the fact (4.26), we get the conclusion (4.24) that is somewhat like

$$\text{"RED"} \Rightarrow \text{"SWEET"}. \tag{4.28}$$

Note that the conclusion (4.27) is not valuable in the market. What we want in the market is the conclusion such as (4.28) (or (4.24)).

Example 4.15. (*Continued from Example* 4.14). Instead of (4.22), assume that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{12}, S_{\delta_{\omega_n}})}{\Longleftarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}(\mathbf{O}_{23}, S_{\delta_{\omega_n}})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{4.29}$$

Using Notation (4.23). When we observe that the tomato ω_n is "RED", we can infer, by the fuzzy inference $\mathbf{M}(\mathbf{O}_{13}, S_{\delta_{\omega_n}})$, the probability that the tomato ω_n is "SWEET" is given by

$$Q=\frac{[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)}{[F_{13}(\{y_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)+[F_{13}(\{n_{\rm SW}\}\mathsf{X}\{y_{\rm RD}\})](\omega_n)} \tag{4.30}$$

which is, by (4.17), estimated as follows:

$$\begin{aligned}&\max\left\{\frac{[F_{\rm RP}(\{y_{\rm RP}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}, \frac{[F_{\rm SW}(\{y_{\rm SW}\})]+[F_{\rm RD}(\{y_{\rm RD}\})]-1}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}\right\}\\ &\quad\leqslant Q\leqslant \min\left\{\frac{[F_{\rm SW}(\{y_{\rm SW}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}, 1\right\}.\end{aligned} \tag{4.31}$$

Note that (4.29) respectively implies (and is implied by)

$$\text{"RIPE"} \Rightarrow \text{"SWEET"} \quad \text{and} \quad \text{"RIPE"} \Rightarrow \text{"RED"}. \tag{4.32}$$

And note that the conclusion (4.31) is somewhat like

$$\text{"RED"} \Rightarrow \text{"SWEET"}. \tag{4.33}$$

Therefore, this conclusion is peculiar to "'fuzziness".

The following theorem is a generalization of the first part of Theorem 4.12.

Theorem 4.16 (Fuzzy syllogism or standard syllogism). *Using Notation* 4.8. *Let* $\delta_{\omega_0} \in \mathscr{M}^p_{+1}(\Omega)$. *Assume that*

$$\mathbf{O}_k^{\{y_k\}} \underset{\mathbf{M}(\mathbf{O}_{k,k+1}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_{k+1}^{\{y_{k-1}\}} \qquad (\forall k = 1, 2, \ldots, |K| - 1), \tag{4.34}$$

Let $\mathbf{O}_K$ *be any observable as in Notation* 4.8, *i.e.*, $\mathscr{G} = [\mathbf{O}_{12}, \mathbf{O}_{23}, \mathbf{O}_{34}, \ldots, \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K$. *Put* $\mathbf{O}_{1,|K|} = \mathbf{O}_K|_{\{1,|K|\}}$. *Then, we see that*

$$\mathrm{Rep}[\mathbf{O}_{1,|K|}]_{at\ \omega_0} = \begin{bmatrix} p^{11}_{1,|K|}(\omega_0) & p^{10}_{1,|K|}(\omega_0) \\ p^{01}_{1,|K|}(\omega_0) & p^{00}_{1,|K|}(\omega_0) \end{bmatrix} = \begin{bmatrix} p^1_1(\omega_0) & 0 \\ p^1_{|K|}(\omega_0) - p^1_1(\omega_0) & 1 - p^1_{|K|}(\omega_0) \end{bmatrix}$$

hence, we see that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_{1,|K|}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_{|K|}^{\{y_{|K|}\}}. \tag{4.35}$$

Proof. Let $\mathbf{O}_K$ be any observable such that $\mathscr{G} = [\mathbf{O}_{12}, \mathbf{O}_{23}, \mathbf{O}_{34}, \ldots, \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K$. Thus, we see that $[\mathbf{O}_K|_{\{1,3\}}, \mathbf{O}_{34}, \ldots, \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K|_{K\setminus\{2\}}$. Note that $(\mathbf{O}_K|_{\{1,3\}})|_{\{m\}} = \mathbf{O}_m$, $m = 1, 3$. Also note, by (4.14), that

$$\mathrm{Rep}[\mathbf{O}_K|_{\{1,3\}}]_{\mathrm{at}\ \omega_0} = \begin{bmatrix} p^1_1(\omega_0) & 0 \\ p^1_3(\omega_0) - p^1_1(\omega_0) & 1 - p^1_3(\omega_0) \end{bmatrix},$$

and therefore

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}(\mathbf{O}_K|_{\{1,3\}}, S_{\delta_{\omega_0}})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}.$$

Hence, by induction, we see that $\mathrm{Rep}[\mathbf{O}_{1,|K|}] \equiv \mathrm{Rep}[\mathbf{O}_K|_{\{1,|K|\}}] = (4.35)$ at $\omega = \omega_0$. This completes the proof. □

Remark 4.17. Everyone knows the fact (i.e., (4.34)⇒(4.36)) in this theorem. However, one must not assume that this theorem is trivial. In fact, this is remarkable as it gives the answer to the question: "Why is the standard syllogism applicable to our life in the real world?" That is, Theorem 4.16 guarantees the justification of the standard syllogism for classical systems. In other words, the obvious fact (i.e., (4.34)⇒(4.36)) is one of tests for our axiom. Also note, as stated before (cf. Remark 4.13), that this "obvious" fact is meaningless in quantum systems. Clearly logic has no ability to guarantee the justification of syllogisms for actual systems because logic is made as a symbolization of "obvious facts" (cf. Remark 3.7(ii)). (We of course know the importance of logic as the foundation of mathematics.) Thus, in the light of our theory, for the first time we can deeply appreciate actual syllogisms.

Remark 4.18. Since our axiom should be the principle that dominates all measurements (i.e., all inferences), it naturally includes some actual "logic". For example, (i) "usual logic" for $\mathscr{A} = C(\Omega)$ and crisp observables, (ii) "quantum logic" for $\mathscr{A} = \mathscr{C}(V)$ and crisp observables, (iii) "fuzzy logic" for $\mathscr{A} = C(\Omega)$ and fuzzy observables. (We expect that some other "logic" will be found in our axiom.) Since our mind is the same as physicists' (and not logicians') in this paper, we did not intend to make "symbolic logic" as a mathematical theory (cf. Remark 3.7(ii)). However, we believe, from the practical point of view, that symbolic logic is often beneficial even if it leaves off reality (i.e., the most important concept "measurement" disappears). That is because it may be used as a handy language (for example, computer language), which is convenient to get quick and general conclusions about particular problems.

Lastly let us compare our theory with other "fuzzy" theories. We have an opinion that Born's axiom is the only one authorized theory of all other "fuzzy" theories (in the wide sense) such as Zadeh's fuzzy theory, Kolmogorov's probability theory, bayesian statistics and so on. In fact, none (but very few heretics) cries against Born's axiom. Quantum mechanics (="Born's axiom" + "Heisenberg's kinetic equation") is surely the most successful theory of this century. Note that Kolmogorov's theory is "mathematics", as he himself said (and gave up proposing the principle) in his famous book [10]. Compared to Born's axiom, others do not have firm foundations (i.e., principles such as Axioms 0 and 1). Therefore, they may be regarded as "mathematics" or "useful method". In fact, only Born's axiom can be judged by "true or not true", others are estimated by "useful or not useful" (or "from the mathematical point of view"). None but Born could propose the principle because he alone found out the importance of "measurement". (This may be due to his genius and the moderate difficulty of quantum mechanics.) Thus, as "objective theory", only Born's axiom is worth believing in. Therefore, we have a reason to assert the justification of our theory.

As stated everywhere in this paper, our proposal (as well as Born's axiom) is objective, that is, " objective fuzzy theory". It is a fact of course, that a principle (i.e., "objective theory") should be unique. Therefore, if our proposal (i.e., Axioms 0 and 1) should include something superfluous (i.e., even a little mathematical generalization), it should be regarded as false (or, "method", "mathematics"). On the other hand, we can possess a lot of "subjective (or mathematical) fuzzy theories". If one wants a "subjective (or mathematical) fuzzy theory", one can make a lot of "subjective (or mathematical) fuzzy theories" as a mathematical or technical generalization of a certain aspect of our axiom. If the proposal is useful for a certain purpose, it should be estimated as a good "theory" (or "method").

Our sketch concerning the above arguments is as follows. For example, from the hint of Remark 3.3(i), one may make a useful "theory" (i.e., membership function method). His proposal is surely useful for computer science. Also, another may start from a probability space $(X, \mathscr{P}(X), \mu)$ in Remark 2.5. This probability space is applicable to both "objective" and "subjective" problems. In this sense, it is quite useful. One will also find some resemblance between Axiom $1''$ and Bayes' theorem. However, only when one explicitly shows the correspondence "$\mathscr{M}(\mathscr{C}_X, S_\Theta) \mapsto (X, \mathscr{P}(X), \mu)$", his assertion is objective. Also, according to Remark 4.18, logicians may make some practical logic. (For the precise observations concerning this sketch, see [8].)

Though this sketch must be examined from various points of view, we believe that our theory is quite general as in the above sketch. However, the most important assertion in this paper is not "generality" but "objectivity". That is because we can easily make several general mathematical theories.

5. Conclusions

In this paper we proposed a foundation of fuzzy measurement theory (i.e., " objective fuzzy theory"). We must emphasize the importance of "measurement". Without the concept of "measurement", none can assert an objective statement concerning "fuzziness" (i.e., Axioms 0 and 1). We can easily expect that our theory is quite applicable because Born's axiom has been so for quantum systems. Here we also proposed the identification: "measurement" = "inference", therefore,their mathematical representations are the same. As one of the applications, we showed and proved several fuzzy syllogisms for classical systems. These results are remarkable because they have realities. We believe in the objectivity of our theory. However, in order that our proposal is completely authenticated, our axiom must be examined from various points of view. Also we have an opinion that any "subjective (or mathematical) fuzzy theory" is characterized as a certain aspect of our axiom. We believe that our proposal is a straightforward approach to "fuzziness".

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